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ANTISYMMETRIC VIBRATIONS OF MULTILAYERED CONICAL SHELLS WITH CONSTRAINED VISCOELASTIC LAYERS

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Abstract—The governing equations of motion for antisymmetric vibrations of a general multilayered conical shell consisting of an arbitrary number of specially orthotropic material layers have been derived using the variational principles. The analysis considers bending, extension, in-plane shear and transverse shear deformations in each of the layers and also includes its rotary, longitudinal translatory and transverse inertias. The Galerkin method has been applied for finding the approximate solution of the shell. The correspondence principle of linear viscoelasticity has been used for evaluating the damping effectiveness of the shells consisting of elastic and viscoelastic layers. A computer program has been developed for antisymmetric vibrations of a general multilayered conical shell consisting of an arbitrary number of elastic and viscoelastic layers. Variation of resonance frequencies and the associated system loss factors with shear parameter and thickness ratio parameter for all families of modes of vibration has been reported for three-, five- and seven-layered conical shells with three sets of classical end conditions: simply supported at both ends, clamped–clamped and free–free. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The multilayered conical shell construction is widely used in aircraft, missile, and spacecraft structures. A number of papers have been published on the vibration and damping analysis of beams, plates and shells in constrained or unconstrained arrangement. Most of these works have been reviewed by Nakra (1984). However, not much work has been done on the vibration of multilayered conical shells. A complete survey of the work up to 1964, on static and dynamic analysis of sandwich structures has been reported by Habip (1965). Extensive review work on the vibration of shells has been reported by Bert and Egle (1969) and Leissa (1973). Bert and his team of researchers have reported an exhaustive work on free vibrations of layered shells [e.g. Bacon and Bert (1967), Bert and Ray, (1969), Siu and Bert (1970), Wilkins *et al.* (1970)]. Bacon and Bert (1967) presented a general method for determining the vibrational behaviour of arbitrary open-ended sandwich shells of revolution, using Rayleigh–Ritz technique and reported numerical results for the case of freely supported edges. Love's (1944) first approximation shell theory with transverse shear deformation and added sandwich effects has been used and all components of translational and rotary inertias have been included in their analysis. Double curved shells of revolution, in their work, have been approximated by a finite number of truncated conical shell elements. Wilkins *et al.* (1970) presented an analysis of free vibrations of sandwich conical shells, considering facings and core of orthotropic materials and all components of translational and rotary inertias were included. In their investigation, the core was capable of resisting transverse shear, but not bending, extension or in-plane shear, and the facings resisted extension, bending and transverse and in-plane shear. The material damping was neglected. In their analysis, the Galerkin method was used for finding the solution of the shell with various boundary conditions. Another analysis for free vibrational modes of sandwich conical shells with free edges has been presented by Siu and Bert (1970). Using Rayleigh–Ritz method, they reported numerical results and compared these with experimental data for conical frustum shells of homogeneous isotropic material and of sandwich construction with specially orthotropic facings and core. In the analysis presented by Chandrasekaran and Ramamurti (1982) for free vibrations of layered truncated conical shell with various boundary conditions, rotary inertia has been neglected and individual layers in the shell have been made of special orthotropic material. They reported results for two-layered conical shell using two methods—finite element method and Rayleigh–Ritz method and also verified the results experimentally.

A literature survey of the field of dynamic analysis of conical shells indicates that no well-documented work is available on vibration and damping analysis of multilayered conical shell with elastic and viscoelastic layers, in which transverse shear deformation, along with extension and bending of layers and transverse, rotary and longitudinal translatory inertias of the shell, have also been included. The present work is an effort in this direction.

In the present paper, the governing equations of motion for vibrations of a general multilayered conical shell, having an arbitrary number of orthotropic material layers, have been derived using the variational principles. In the present analysis, extension, bending, in-plane shear and transverse shear in all the layers have been considered, and transverse, longitudinal translatory, and rotary inertias are taken into account. Love's first approximation shell theory with transverse shear strain added is used and solutions are obtained by Galerkin's method. The correspondence principle of linear viscoelasticity has been used for evaluating the damping effectiveness of conical shells with elastic and viscoelastic layers. A computer program for determining the resonance frequencies and the associated system loss factors for various families of modes of antisymmetric vibrations of a general multilayered conical shell consisting of an arbitrary number of specially orthotropic elastic and viscoelastic layers has been developed. The program has been validated for the vibration of elastic two- and three-layered conical shells (Wilkins *et al.*, 1970; Chandrasekaran and Ramamurti, 1982), and also for the resonance frequencies and the associated system loss factors of the multilayered cylindrical shells (Alam and Asnani, 1984). The variations of the resonance frequencies and the associated system loss factors for various families of modes for antisymmetric vibrations of multilayered conical shells consisting of alternate elastic and viscoelastic layers with the shear parameter and the thickness ratio parameter, are reported for various edge conditions.

2. GOVERNING EQUATIONS OF MOTION

The cross-section of an N-layered truncated conical shell is shown in Fig. 1. The curvilinear coordinate system is employed with displacements u, v, and w in X, ϕ, and Z directions, respectively. The assumed deformation patterns in the circumferential and the meridional directions have also been shown in the figure. It is assumed that the deflections are small and the material of the layers is specially orthotropic. The normal cross-sections

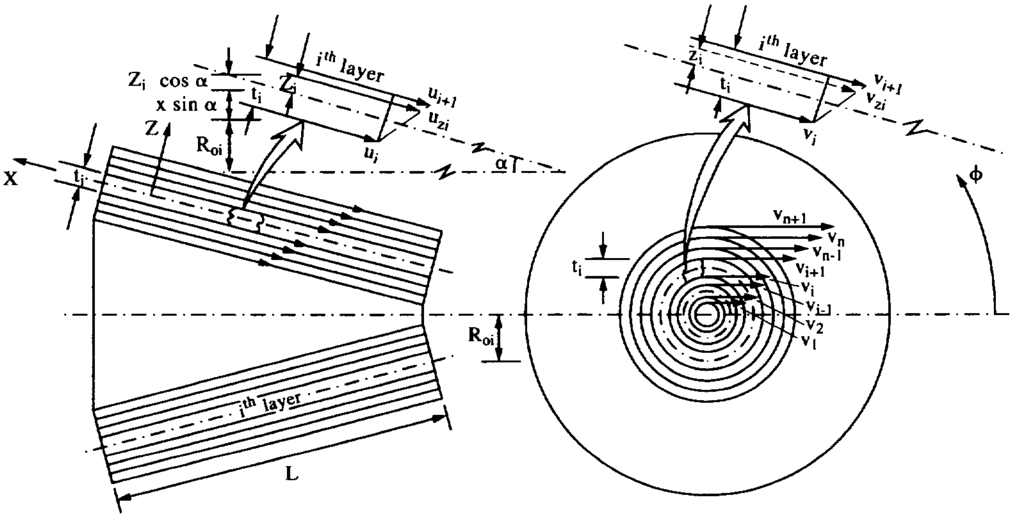

Fig. 1. Assumed deformation pattern in the miltilayered conical shell.

in all the layers are assumed to remain plane and continuous before and after deformations. The deformations in the layers take account of bending, extension, in-plane shear and transverse shear. It is assumed that there is no interface slip between the layers.

The deformations u_{zi} and v_{zi} in the ith layer along X and ϕ directions at a distance z_i from the middle of this layer are given as

$$u_{zi} = \frac{1}{t_i}\left[u_i\left(\frac{t_i}{2}-z_i\right)+u_{i+1}\left(\frac{t_i}{2}+z_i\right)\right], \tag{1}$$

$$v_{zi} = \frac{1}{t_i}\left[v_i\left(\frac{t_i}{2}-z_i\right)+v_{i+1}\left(\frac{t_i}{2}+z_i\right)\right].$$

The strain components for the curvilinear coordinate system in the ith layer of shell are (Love, 1944)

$$\begin{aligned}
(\varepsilon_{xx})_i &= u_{zi,x}\\
(\varepsilon_{\phi\phi})_i &= \frac{1}{r_i}(v_{zi,\phi}+u_{zi}\sin\alpha+w\cos\alpha)\\
(\varepsilon_{zz})_i &= w_{,x}\\
(\gamma_{\phi z})_i &= \frac{1}{r_i}(w_{,\phi}-v_{zi}\cos\alpha)+v_{zi,z}\\
(\gamma_{zx})_i &= w_{,x}+u_{zi,z}\\
(\gamma_{x\phi})_i &= \frac{1}{r_i}(u_{zi,\phi}-v_{zi}\sin\alpha)+v_{zi,x}
\end{aligned} \tag{2}$$

where $r_i = (R_{0i}+x\sin\alpha+z_i\cos\alpha)$ and u_{zi}, v_{zi}, and w are displacements in the ith layer of the shell in X, ϕ, and Z directions, respectively, and subscripts $_{,x}$, $_{,\phi}$ and $_{,z}$ denote the partial derivatives with respect to x, ϕ, and z, respectively.

Let

$$\zeta_i = \frac{1}{r_i} = (R_{0i}+x\sin\alpha+z_i\cos\alpha)^{-1}. \tag{3}$$

On substitution of eqns (1) into eqns (2) and using eqn (3), the expressions become

$$\begin{aligned}
(\varepsilon_{xx})_i &= \frac{1}{t_i}\left[u_{i,x}\left(\frac{t_i}{2}-z_i\right)+u_{i+1,x}\left(\frac{t_i}{2}+z_i\right)\right]\\
(\varepsilon_{\phi\phi})_i &= \frac{\zeta_i}{t_i}\left[(v_{i,\phi}+u_i\sin\alpha)\left(\frac{t_i}{2}-z_i\right)+(v_{i+1,\phi}+u_{i+1}\sin\alpha)\left(\frac{t_i}{2}+z_i\right)+t_iw\cos\alpha\right]\\
(\varepsilon_{zz})_i &= 0\\
(\gamma_{\phi z})_i &= \frac{\zeta_i}{t_i}\left[t_iw_{,\phi}-v_i\cos\alpha\left(\frac{t_i}{2}-z_i\right)-v_{i+1}\cos\alpha\left(\frac{t_i}{2}+z_i\right)\right]+\frac{(v_{i+1}-v_i)}{t_i}\\
(\gamma_{zx})_i &= w_{,x}+\frac{(u_{i+1}-u_i)}{t_i}
\end{aligned}$$

$$(\gamma_{x\phi})_i = \frac{\zeta_i}{t_i}\left[(u_{i,\phi} - v_i \sin\alpha)\left(\frac{t_i}{2} - z_i\right) + (u_{i+1,\phi} - v_{i+1}\sin\alpha)\left(\frac{t_i}{2} + z_i\right)\right]$$

$$+ \frac{1}{t_i}\left[v_{i,x}\left(\frac{t_i}{2} - z_i\right) + v_{i+1,x}\left(\frac{t_i}{2} + z_i\right)\right]. \quad (4)$$

Strain energy U of the shell is given by

$$U = \frac{1}{2}\sum_{i=1}^{N}\int_x\int_\phi\int_z [(\sigma_{xx})_i(\varepsilon_{xx})_i + (\sigma_{\phi\phi})_i(\varepsilon_{\phi\phi})_i + (\tau_{\phi z})_i(\gamma_{\phi z})_i + (\tau_{zx})_i(\gamma_{zx})_i$$

$$+ (\tau_{x\phi})_i(\gamma_{x\phi})_i]\zeta_i^{-1}\,\mathrm{d}z\,\mathrm{d}\phi\,\mathrm{d}x. \quad (5)$$

The material of each layer of the shell is considered to be specially orthotropic, and as such has six independent elastic constants that are pertinent. For the ith layer, these elastic constants are $(E_x)_i$, $(E_\phi)_i$, $(G_{\phi z})_i$, $(G_{zx})_i$, $(G_{x\phi})_i$ and $(\nu_{x\phi})_i$.

The stress-strain relationship for each layer can be expressed as

$$\begin{bmatrix}(\sigma_{xx})_i\\(\sigma_{\phi\phi})_i\\(\tau_{x\phi})_i\end{bmatrix} = \begin{bmatrix}(Q_{11})_i & (Q_{12})_i & 0\\(Q_{21})_i & (Q_{22})_i & 0\\0 & 0 & (Q_{66})_i\end{bmatrix}\begin{bmatrix}(\varepsilon_{xx})_i\\(\varepsilon_{\phi\phi})_i\\(\gamma_{x\phi})_i\end{bmatrix}$$

$$\begin{bmatrix}(\tau_{\phi z})_i\\(\tau_{xz})_i\end{bmatrix} = \begin{bmatrix}(C_{44})_i & 0\\0 & (C_{55})_i\end{bmatrix}\begin{bmatrix}(\gamma_{\phi z})_i\\(\gamma_{xz})_i\end{bmatrix} \quad (6)$$

where the material constants are

$$(Q_{11})_i = \frac{(E_x)_i}{[1-(\nu_{x\phi})_i(\nu_{\phi x})_i]}$$

$$(Q_{22})_i = \frac{(E_\phi)_i}{[1-(\nu_{x\phi})_i(\nu_{\phi x})_i]}$$

$$(Q_{12})_i = \frac{(\nu_{\phi x})_i(E_x)_i}{[1-(\nu_{x\phi})_i(\nu_{\phi x})_i]}$$

$$(Q_{21})_i = \frac{(\nu_{x\phi})_i(E_\phi)_i}{[1-(\nu_{x\phi})_i(\nu_{\phi x})_i]}$$

$$(\nu_{\phi x})_i(E_x)_i = (\nu_{x\phi})_i(E_\phi)_i \quad \text{making } (Q_{12})_i = (Q_{21})_i$$

$$(Q_{66})_i = (G_{x\phi})_i; \quad (C_{44})_i = (G_{\phi z})_i; \quad (C_{55})_i = (G_{xz})_i. \quad (7)$$

Using eqns (4) and (7) and performing the integration over Z, the expression of the strain energy U may be obtained [see Appendix, eqn (A1)].

The kinetic energy of the multilayered conical shell is given by

$$T = \frac{1}{2}\sum_{i=1}^{N}\int_x\int_\phi\int_z [\rho_i(\dot{w}^2 + \dot{u}_{zi}^2 + \dot{v}_{zi}^2)]\zeta_i^{-1}\,\mathrm{d}z\,\mathrm{d}\phi\,\mathrm{d}x \quad (8)$$

where ($\dot{}$) denotes the differentiation with respect to time.

Using eqn (1) and performing integration over Z, the kinetic energy may be obtained [see Appendix, eqn (A3)].

The work done by the external excitation forces $f(x, \phi)\, g(t)$ is given by

$$W = \int_x \int_\phi f(x, \phi) g(t) w \, \mathrm{d}\phi \, \mathrm{d}x. \tag{9}$$

Performing the variation term by term and making use of Hamilton's principle, the governing equations of motion and boundary conditions are obtained [see Appendix: eqns (A4)–(A6)].

3. SOLUTION FOR ANTISYMMETRIC VIBRATIONS OF MULTILAYERED CONICAL SHELLS

The Galerkin method has been applied for finding the approximate solution with various boundary conditions at edges, i.e. simply supported edges, clamped edges and free edges by assuming suitable solution functions.

The simple support boundary condition at the edges is defined here as zero displacements in the circumferential and transverse directions and the unrestricted displacement in the meridional direction, and so the assumed solution functions have been taken as

$$U_i = \sum_{m=1}^{\infty} U_{i,mJ} \cos\frac{m\pi x}{L} \sin J\phi \sin \omega t$$

$$V_i = \sum_{m=1}^{\infty} V_{i,mJ} \sin\frac{m\pi x}{L} \cos J\phi \sin \omega t$$

$$W = \sum_{m=1}^{\infty} W_{,mJ} \sin\frac{m\pi x}{L} \sin J\phi \sin \omega t \tag{10}$$

where $m = 1, 2, 3, 4, \ldots, n$ and $J = 1, 2, 3, 4, \ldots$

The clamped–clamped boundary condition at the edges is defined here as zero displacements in the circumferential, transverse, and meridional directions, and so the assumed solution functions have been taken as

$$U_i = \sum_{m=1}^{\infty} U_{i,mJ} \sin\frac{m\pi x}{L} \sin J\phi \sin \omega t$$

$$V_i = \sum_{m=1}^{\infty} V_{i,mJ} \sin\frac{m\pi x}{L} \sin J\phi \sin \omega t$$

$$W = \sum_{m=1}^{\infty} W_{,mJ} \sin\frac{m\pi x}{L} \sin J\phi \sin \omega t \tag{11}$$

where $m = 1, 2, 3, 4, \ldots, n$ and $J = 1, 2, 3, 4, \ldots$

For the free edge boundary condition, the forces and the moments must be zero. It is very difficult to find a set of simple trigonometric series which will satisfy the involved differential equations of these boundary conditions. From the fact that the displacements and the rotations at the free edges will always be unrestrained, the series of cosine functions having non-zero values at the edges are taken as solution functions,

$$U_i = \sum_{m=1}^{\infty} U_{i,mJ} \cos\frac{m\pi x}{L} \sin J\phi \sin \omega t$$

$$V_i = \sum_{m=1}^{\infty} V_{i,mJ} \cos\frac{m\pi x}{L} \cos J\phi \sin \omega t$$

$$W = \sum_{m=1}^{\infty} W_{,mJ} \cos\frac{m\pi x}{L} \sin J\phi \sin \omega t \tag{12}$$

where $m = 0, 1, 2, 3, 4, \ldots, n$ and $J = 1, 2, 3, 4, 5, \ldots$

The excitation may be expanded as

$$F = \sum_{m=1}^{\infty} F_{mJ} \sin\frac{m\pi x}{L} \sin J\phi \sin \omega t. \tag{13}$$

A set of error functions is generated by the substitution of the solution functions [the assumed displacement equations (10) or (11) or (12) as the case may be] in the governing differential equations of motion (A4)–(A6). These error functions are weighted by assumed solution functions (coefficients of the displacements $U_{i,1J}$, $U_{i,2J}$, $U_{i,3J}, \ldots$, $U_{i,mJ}$, $V_{i,1J}$, $V_{i,2J}$, $V_{i,3J}, \ldots$, $V_{i,mJ}$, W_{1J}, W_{2J}, $W_{3J}, \ldots$, W_{mJ}). All these weighted errors, integrated over the entire conical shell, are set equal to zero. The integrals with respect to coordinate X over the slant length L are evaluated by numerical integration. The four-term (m = 1, 2, 3, 4) solution is taken for shells with simply supported edges and clamped edges, and five-term (m = 0, 1, 2, 3, 4) solution for shell with free edges. Thus a set of $m(2N+3)$ number of simultaneous algebraic equations are obtained which can be written in standard eigen value form as follows:

$$[\mathbf{A} - \omega^2 \mathbf{B}][\mathbf{X}] = 0 \tag{14}$$

where column vector $[\mathbf{X}] = [U_{1,1J}, U_{2,1J}, U_{3,1J}, \ldots, U_{(N+1),1J}, U_{1,2J}, U_{2,2J}, U_{3,2J}, \ldots, U_{(N+1),2J}, \ldots, U_{1,mJ}, U_{2,mJ}, U_{3,mJ}, \ldots, U_{(N+1),mJ}, V_{1,1J}, V_{2,1J}, V_{3,1J}, \ldots, V_{(N+1),1J}, V_{1,2J}, V_{2,2J}, V_{3,2J}, \ldots, V_{(N+1),2J}, \ldots, V_{1,mJ}, V_{2,mJ}, V_{3,mJ}, \ldots, V_{(N+1),mJ}, W_{1J}, W_{2J}, W_{3J}, \ldots, W_{mJ}]^T$

$\mathbf{A}$ and $\mathbf{B}$ are square matrices of order $m(2N+3)$. The elements of these matrices are the functions of the geometric and material properties of the shell. Replacing the real moduli by the complex moduli according to the correspondence principle of linear viscoelasticity for harmonic motion, the coefficients of the terms of the above sets of simultaneous algebraic equations will become complex. After transforming the set of equations into standard eigen value form, all eigen values of the matrix are evaluated. The eigen values ω^2 give the resonance frequencies and the associated system loss factors for the coupled modes of the antisymmetric vibrations of the shell. The real part of the eigen value is the square of the resonance frequency in radians per second, and the ratio of the imaginary part to the real part of the eigen value is the associated system loss factor η_s (Rao and Nakra, 1974). It has been shown that η_s is the ratio of the imaginary to the real part of the generalised complex stiffness and also the ratio of energy dissipated per cycle to the maximum strain energy during a cycle (Ungar and Kerwin, 1962).

The above procedure has been programmed to compute the resonance frequencies and the associated system loss factors for all the modes of families of modes of antisymmetric vibrations and the corresponding modal vectors of a multilayered conical shell.

For a general N-layered shell, there are $m(2N+3)$ families of coupled modes. These are identified according to the predominant displacements. For a given value of J, $m(2N+3)$ values of ω are obtained. The displacement ratios

$$\frac{U_{1,mJ}}{W_{mJ}}, \frac{U_{2,mJ}}{W_{mJ}}, \frac{U_{3,mJ}}{W_{mJ}}, \ldots, \frac{U_{(N+1),mJ}}{W_{mJ}}, \frac{V_{1,mJ}}{W_{mJ}}, \frac{V_{2,mJ}}{W_{mJ}}, \frac{V_{3,mJ}}{W_{mJ}}, \ldots, \frac{V_{(N+1),mJ}}{W_{mJ}}$$

are computed from the solution of simultaneous equations. Corresponding to Mode I (radial), values of the displacement ratios are small and this mode is also obtained when only the transverse inertia effect is considered. For families of modes due to (extension+torsion) of the layers of multilayered shell, designated as Modes II and III, the displacement ratios

$$\frac{U_{1,mJ}}{W_{mJ}}, \frac{U_{2,mJ}}{W_{mJ}}, \frac{U_{3,mJ}}{W_{mJ}}, \ldots, \frac{U_{(N+1),mJ}}{W_{mJ}}$$

are of the same sign and so are the displacement ratios

$$\frac{V_{1,mJ}}{W_{mJ}}, \frac{V_{2,mJ}}{W_{mJ}}, \frac{V_{3,mJ}}{W_{mJ}}, \ldots, \frac{V_{(N+1),mJ}}{W_{mJ}}.$$

Families of Modes IV, V, VI and VII are mainly core thickness shear type. A pair or more pairs of elastic layers move in opposite directions causing shear of viscoelastic cores in meridional and circumferential directions. The total number of such modes in N-layered shell are $(N-1)$. In the remaining $(N+1)$ modes, for N-layered shell, some of the displacement ratios

$$\frac{U_{1,mJ}}{W_{mJ}} \text{ and } \frac{U_{2,mJ}}{W_{mJ}}, \frac{U_{3,mJ}}{W_{mJ}} \text{ and } \frac{U_{4,mJ}}{W_{mJ}},$$

etc., are of opposite sign. Similarly

$$\frac{V_{1,mJ}}{W_{mJ}} \text{ and } \frac{V_{2,mJ}}{W_{mJ}}, \frac{V_{3,mJ}}{W_{mJ}} \text{ and } \frac{V_{4,mJ}}{W_{mJ}},$$

etc., are of opposite sign and they are designated as mainly elastic layer thickness shear type.

4. COMPARISON WITH REPORTED RESULTS

The validation of the presently developed analysis has been accomplished by comparing the results obtained by the analysis with the results reported by several investigators (Weingarten, 1965; Siu and Bert, 1970; Wilkins *et al.*, 1970; Chandrasekaran and Ramamurti, 1982; Alam and Asnani, 1984). The natural frequencies of a sandwich conical shell consisting of elastic layers with various end conditions have been determined (Table 1) with the present analysis for data:

Table 1. Comparison with analytical frequencies for simply supported, clamped–clamped and free–free sandwich conical shells reported by Wilkins *et al.* (1970)

	Simply supported		Clamped–clamped		Free–free	
J	$m = 1$	$m = 2$	$m = 1$	$m = 2$	$m = 1$	$m = 2$
2	135.57	311.40	188.83	358.12	12.64	188.70
	(134.8)†	(310.9)	(177.2)	(340.1)	(13.3)	(179.9)
3	85.88	213.27	135.65	278.11	33.98	135.78
	(86.8)	(212.8)	(126.0)	(254.3)	(35.0)	(130.2)
4	84.22	165.78	117.08	239.48	64.42	117.98
	(85.7)	(165.4)	(110.7)	(209.7)	(65.3)	(115.7)
5	112.17	160.91	130.44	231.27	101.43	134.10
	(113.3)	(160.6)	(126.7)	(197.7)	(101.8)	(133.8)
6	153.21	189.02	166.55	247.78	143.74	175.60
	(153.7)	(188.9)	(163.5)	(214.8)	(143.3)	(175.6)
7	201.72	236.68	216.45	284.22	192.61	231.30
	(201.5)	(236.5)	(212.3)	(254.5)	(191.1)	(229.7)
8	257.04	294.96	275.62	336.46	248.32	295.44
	(256.2)	(294.1)	(269.1)	(310.0)	(245.7)	(290.5)
9	318.98	361.18	342.08	401.40	310.48	365.94
	(317.3)	(358.7)	(332.7)	(376.1)	(306.7)	(357.8)

† Values in parentheses are from table reported by Wilkins *et al.* (1970).

$$\alpha = 5.07^\circ,$$
$$L = 72.5\,\text{in},$$
$$R_{01} = 22.290\,\text{in},$$
$$R_{02} = 22.450\,\text{in},$$
$$R_{03} = 22.609\,\text{in},$$
$$t_1 = 0.021\,\text{in},$$
$$t_2 = 0.3\,\text{in},$$
$$t_3 = 0.021\,\text{in},$$
$$E_{x1} = E_{x3} = E_{\phi 1} = E_{\phi 3} = 3.64 \times 10^6\,\text{lb in}^{-2},$$
$$G_{zx2} = 3.2 \times 10^4\,\text{lb in}^{-2},$$
$$G_{\phi z2} = 1.83 \times 10^4\,\text{lb in}^{-2},$$
$$G_{zx1} = G_{zx3} = G_{\phi z1} = G_{\phi z3} = G_{x\phi 1} = G_{x\phi 3} = 1.0 \times 10^6\,\text{lb in}^{-2},$$
$$\nu_{x\phi 1} = \nu_{x\phi 3} = \nu_{\phi x1} = \nu_{\phi x3} = 0.2,$$
$$\rho_1 = \rho_3 = 0.265 \times 10^{-3}\,\text{lb s}^2\,\text{in}^{-4},$$
$$\rho_2 = 0.3368 \times 10^{-5}\,\text{lb s}^2\,\text{in}^{-4},$$

where suffix 1 is for inner face layer, 2 for core, and 3 for outer face layer of the sandwich conical shell. For the simply supported case, agreement with the reported results is quite good. Frequencies for the clamped–clamped and free–free sandwich conical shells with the present analysis are found to be somewhat different from those reported by Wilkins *et al.* (1970). This difference in values can be attributed to the difference in boundary conditions in the two analysis.

The resonance frequencies and associated system loss factors for three-layered cylindrical shells (two elastic face layers sandwiching a viscoelastic core) have been computed with the present analysis by taking the cone apex angle α to be zero, and are shown in Fig. 2 and these have been found to be in close agreement with results reported by Alam and Asnani (1984). Also, agreement of results with the ones reported by Weingarten (1965), Siu and Bert (1970) and Chandrasekaran and Ramamurti (1982) is found to be quite satisfactory.

Multilayered conical shell

Multilayered conical shells taken for investigation are cone frustums, consisting of elastic stiff layers sandwiching relatively soft viscoelastic core layers. The shear parameter δ has been defined as the ratio of the in-phase component of the shear modulus ($G_{x\phi}$) of the viscoelastic cores to the Young's modulus (E) of the elastic layers and the thickness ratio parameter V is defined here as the ratio of thickness of the viscoelastic layer to that of the elastic layer. Poisson's ratio (ν) of the elastic material is taken to be 0.3 and the ratio of Poisson's ratio of the viscoelastic material to that of the elastic material is taken to be 1.33. The ratio of mass density (ρ) of viscoelastic to that of elastic material is taken to be 0.5. The loss factor η of the viscoelastic core in shear, as well as in extension, is taken to be 0.5. The other parameters are (i) R_1/L is taken to be 0.1 (here R_1 is the inner radius, and L is the slant length of the shell); (ii) T/R_1 is taken to be 0.5 (here T is the total thickness of the shell); (iii) α, the cone apex angle is taken to be 5°: and (iv) J, the circumferential mode number, is taken to be 1.

5. RESULTS AND DISCUSSION

Parametric studies have been undertaken to provide a complete understanding of the vibration and damping behaviour of three-, five- and seven-layered conical shells consisting

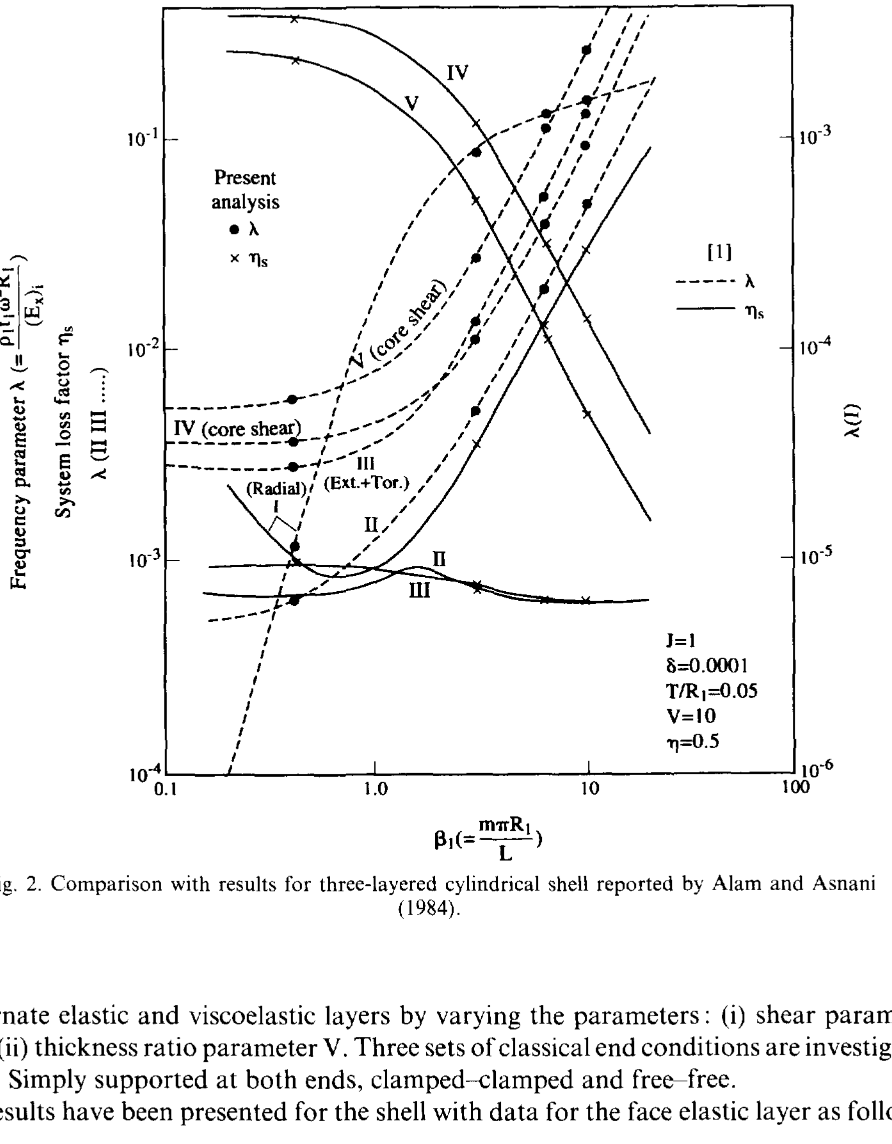

Fig. 2. Comparison with results for three-layered cylindrical shell reported by Alam and Asnani (1984).

of alternate elastic and viscoelastic layers by varying the parameters: (i) shear parameter δ; and (ii) thickness ratio parameter V. Three sets of classical end conditions are investigated herein: Simply supported at both ends, clamped–clamped and free–free.

Results have been presented for the shell with data for the face elastic layer as follows:

Young's modulus: $E_x = E_\phi = 3.64 \times 10^6$ lb in^{-2} = 0.252874×10^{11} N m^{-2},
Shear modulus: $G_{x\phi} = G_{\phi z} = G_{xz} = 1.399994 \times 10^6$ lb in^{-2} = 0.972591×10^{10} N m^{-2},
Density: $\rho = 2.65 \times 10^{-4}$ lb s^2 in^{-4} = 0.28535327×10^4 Ns^2 m^{-4},
Thickness: $t = 0.03$ in = 0.762×10^{-5} m,
Radius: $R_1 = 7.2$ in = 0.18288 m.

In the analysis, the designation m_1 denotes the lowest resonance frequency and its corresponding system loss factor; m_2 denotes the second lowest frequency, etc. For a three-layered sandwich shell, families of modes have been shown for m_1 and m_2, whereas for the sake of clarity in figures for multilayered shells the curves have been drawn for the first lowest frequencies only and their corresponding system loss factors for families of modes.

Figures 3–11 show the variation of resonance frequency ω and associated system loss factor η_s with shear parameter δ for antisymmetric vibration of three-, five- and seven-layered conical shells with simply supported, clamped–clamped and free–free end conditions for $V = 10$.

(a) Effect of δ on multilayered conical shells with simply supported edges

Variation of ω and η_s with δ for multilayered shell with simply supported edges is shown in Figs 3–5.

ω for Mode I (radial) of three-layered sandwich shell remains nearly constant for lower values of δ, it increases marginally at $\delta = 10^{-5}$ and then decreases slightly with further

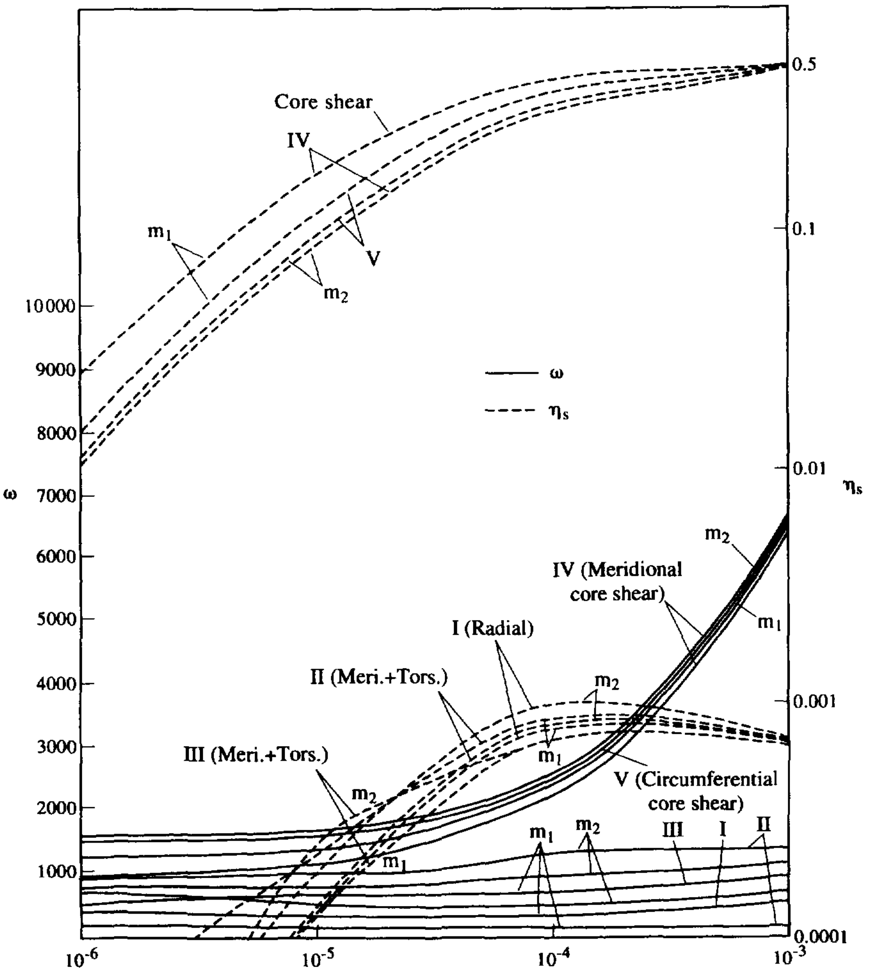

Fig. 3. Variation of ω and η_s with δ for antisymmetric vibration of a simply supported three-layered conical shell.

increase of δ, whereas ω for this mode of five- and seven-layered shell, has no appreciable change for the chosen range of δ. η_s for this mode of sandwich shell increases up to $\delta = 10^{-4}$ and then starts decreasing for further increasing values of δ, whereas η_s for this Mode I for five- and seven-layered shells increases with δ. There is increase in η_s for this mode for a particular value of δ when the number of layers in the shell are increased from three to five, but this increase is only marginal when the number of layers are further increased to seven.

ω for Modes II and III (meridional+torsional) for the multilayered shell have no appreciable change with δ, only a small increase is observed for higher values of δ. η_s for Modes II and III of a three-layered sandwich shell increases in the lower range of δ, reaches a maximum ($= 0.0008$) and then decreases slightly with the higher values of δ. η_s for Mode II for five- and seven-layered shells increases with δ. η_s for Mode III decreases marginally with the increase of δ for five-layered shell whereas for seven-layered shell, it decreases considerably in the lower range of δ, reaches a minimum (0.0035), and then again increases with higher values of δ.

It is thus observed that η_s for Modes II and III (meridional+torsional) is small for a three-layered shell in comparison to the corresponding values for a five-layered shell, specially at a lower values of δ. A seven-layered shell seems to give higher values of η_s for these modes for a certain range of δ and the reverse trend is noticed in some other range of δ.

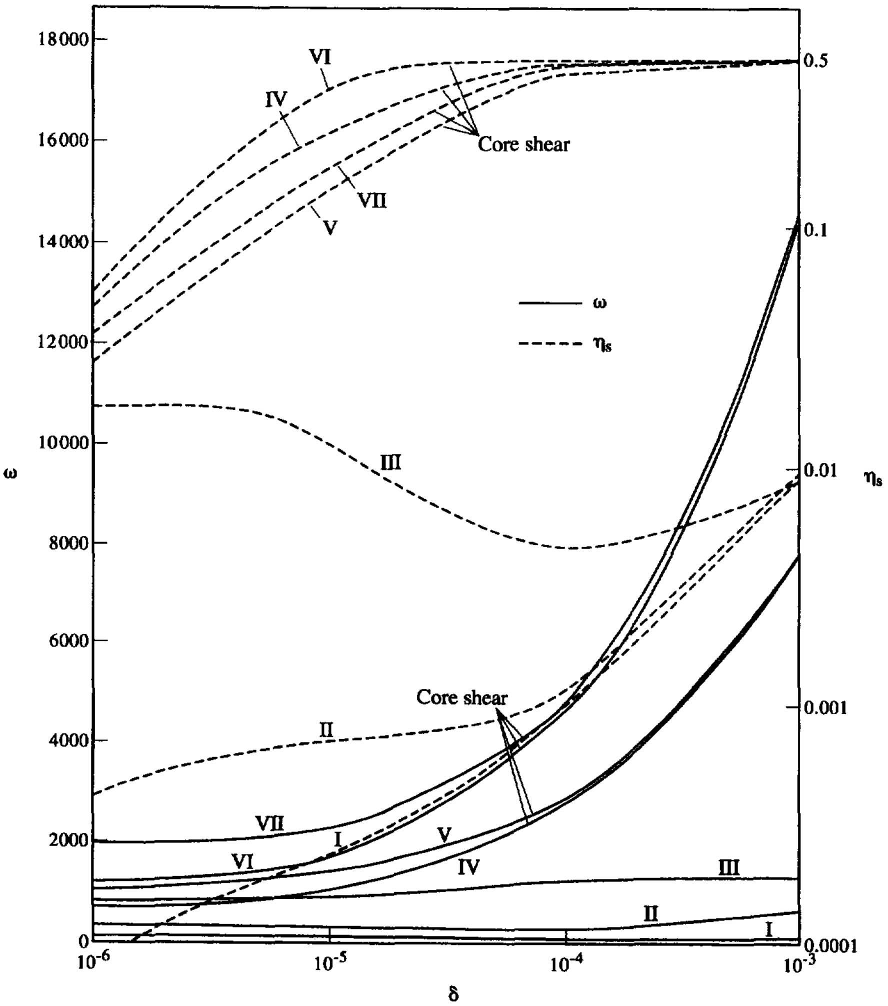

Fig. 4. Variation of ω and η_s with δ for antisymmetric vibration of a simply supported five-layered conical shell.

ω for higher-order modes, due to thickness shear of cores in the multilayered shell, increases with δ. η_s for these modes increases with δ and it also increases with number of layers in the multilayered shell for a particular value of δ.

Thus for getting a uniformly high order of η_s for all families of modes, one should choose higher values of δ and more layers in the multilayered shell.

(*b*) *Effect of* δ *on multilayered conical shells with clamped edges*

Figures 6–8 show the effect of shear parameter δ for multilayered shell with clamped edges.

ω for Mode I of multilayered shell has no appreciable change for the chosen range of δ. η_s for this mode increases with δ for multilayered shell and there is increase in η_s for this mode for a particular value of δ when the number of layers in the shell are increased from three to five, but this increase is observed to be only marginal when the number of layers are further increased to seven.

ω for Modes II and III for the multilayered shell have no appreciable change with δ, only a small increase is observed for higher values of δ. η_s for these modes of three-layered shell increases with the increase of δ. η_s for Mode II of five- and seven-layered shells increases with δ, whereas η_s for Mode III of five- and seven-layered shells decreases in the lower range of δ, reaches a minimum and then starts increasing with further increase of δ.

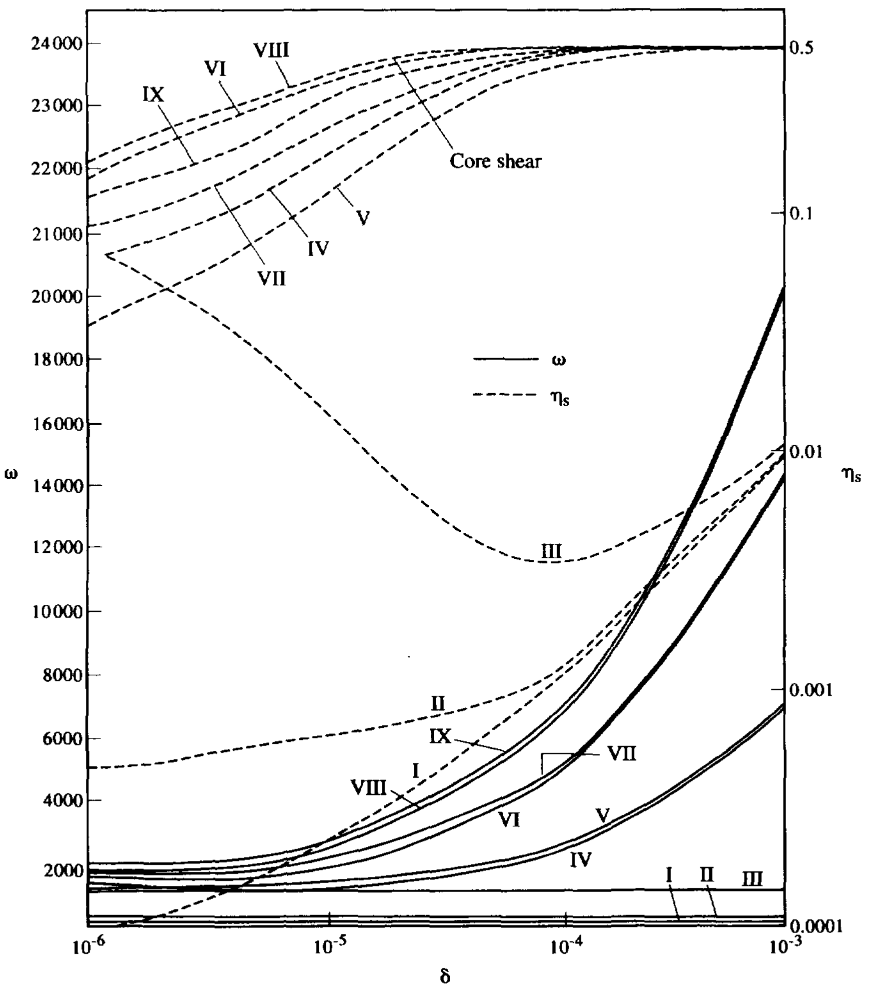

Fig. 5. Variation of ω and η_s with δ for antisymmetric vibration of a simply supported seven-layered conical shell.

ω and η_s for meridional and circumferential core shear modes of multilayered shells have the same trend of variation with δ as those in the case of simply supported shells.

(*c*) *Effect of δ on multilayered conical shells with free edges*

Variation of ω and η_s with δ for multilayered shell with free edges is shown in Figs 9–11.

ω and η_s for Mode I of the free–free multilayered shell follow the same trend of variation with δ as has been observed in the case of shells with clamped edges.

ω for Mode II of three-layered sandwich shell increases with lower values of δ and remains nearly constant in the higher range, whereas ω for Mode III of three-layered shell decreases marginally with δ. ω for Modes II and III for five- and seven-layered shells remains almost constant in the chosen range of δ, only a small increase is noticed for higher values of δ. η_s for these modes of three-layered shell increases with δ whereas η_s for Modes II and III for five- and seven-layered shells decreases in the lower range of δ, reaches a low value in the intermediate values of δ and then starts increasing with further increase of δ.

ω and η_s for higher-order modes, due to thickness shear of core layers in the meridional and circumferential directions, follow the same trend of variation with δ as they have been found to vary in simply supported/clamped–clamped shells.

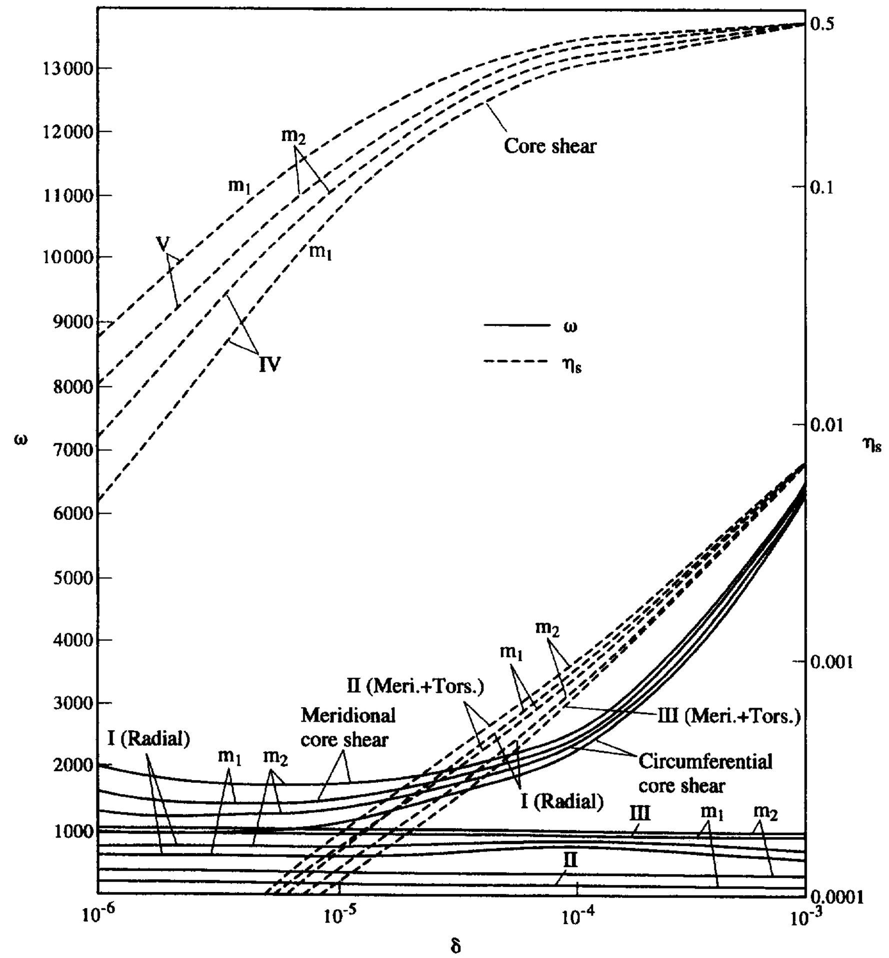

Fig. 6. Variation of ω and η_s with δ for antisymmetric vibration of a clamped–clamped three-layered conical shell.

Variation of ω and η_s with thickness ratio parameter V for antisymmetric vibrations of three-, five- and seven-layered conical shells for three end conditions: simply supported at both ends; clamped–clamped; and free–free for $\delta = 10^{-4}$ are shown in Figs 12–20.

(d) Effect of V *on multilayered conical shells with simply supported edges*

Variation of ω and η_s with thickness ratio parameter V for multilayered shell with simply supported edges is shown in Figs. 12–14.

There is a steep fall in ω for Mode I (radial) for lower values of V (up to $V < 10$) and then a marginal decrease is noticed in the higher range ($V > 10$) for sandwich shell. ω for this mode is high for three-layered sandwich conical shell, specially for lower values of V, as compared with the five-layered shell. ω for Mode I in five- and seven-layered shells increases slightly from $V = 0.5$ to 5 and then a marginal decrease is observed with further increase of V.

ω for Modes II and III (meridional + torsional) in the multilayered shell increases in the lower range of V ($V < 7$) and a decrease is observed with the higher values of V ($V > 7$). This trend of increase/decrease reduces with further increase in number of layers.

η_s for Modes I, II, and III increases with V and there is only a marginal increase in η_s of these modes with more number of layers. Thus for getting a high value of η_s for these modes, one should go for high values of V.

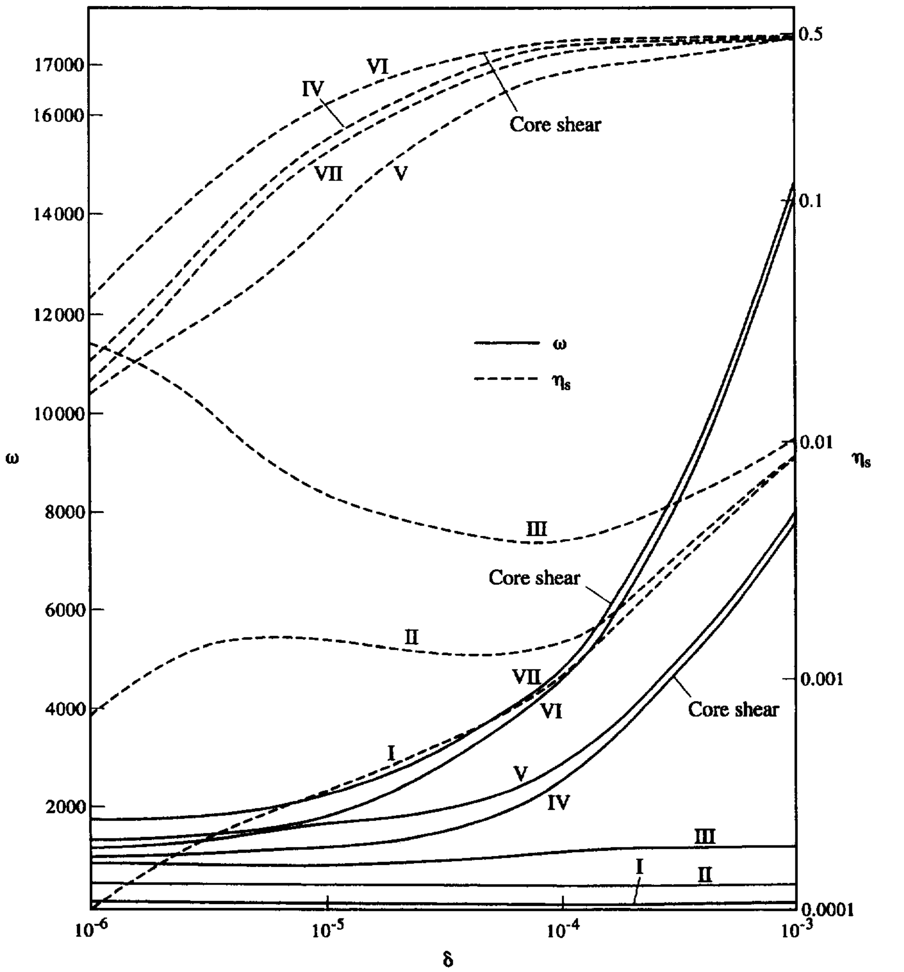

Fig. 7. Variation of ω and η_s with δ for antisymmetric vibration of a clamped–clamped five-layered conical shell.

ω and η_s for meridional and circumferential core shear modes increases with increasing values of V. η_s is more for these modes with more number of layers in the multilayered shell for a particular value of V. Thus one may come to the conclusion that a uniformly high value of η_s for all families of modes are obtained for $V > 10$.

(*e*) *Effect of* V *on multilayered conical shells with clamped edges*

Variation of ω and η_s with V for multilayered shell with clamped edges is shown in Figs 15–17.

ω for Mode I (radial) for multilayered shell follows the same trend of variation with V as has been observed for the case of simply supported shell but η_s for this mode is found to be more than that for the previous case.

ω for Mode II of sandwich shell increases from $V = 0.5$ to 5, starts decreasing for higher values of V (i.e. for $V > 20$). ω for Mode III for three-layered shells increases with lower values of V and decreases in the higher range of V ($V > 10$). Otherwise ω for these Modes II and III for five- and seven-layered shells increases in the lower range of V and a decrease is noticed with the higher values of V. η_s for these modes vary in the same way as has been found for the case of a simply supported case.

Also a similar trend is observed for ω and η_s for higher-order modes due to thickness shear of core layers in the meridional and circumferential directions as is seen for the shell with simply supported edges.

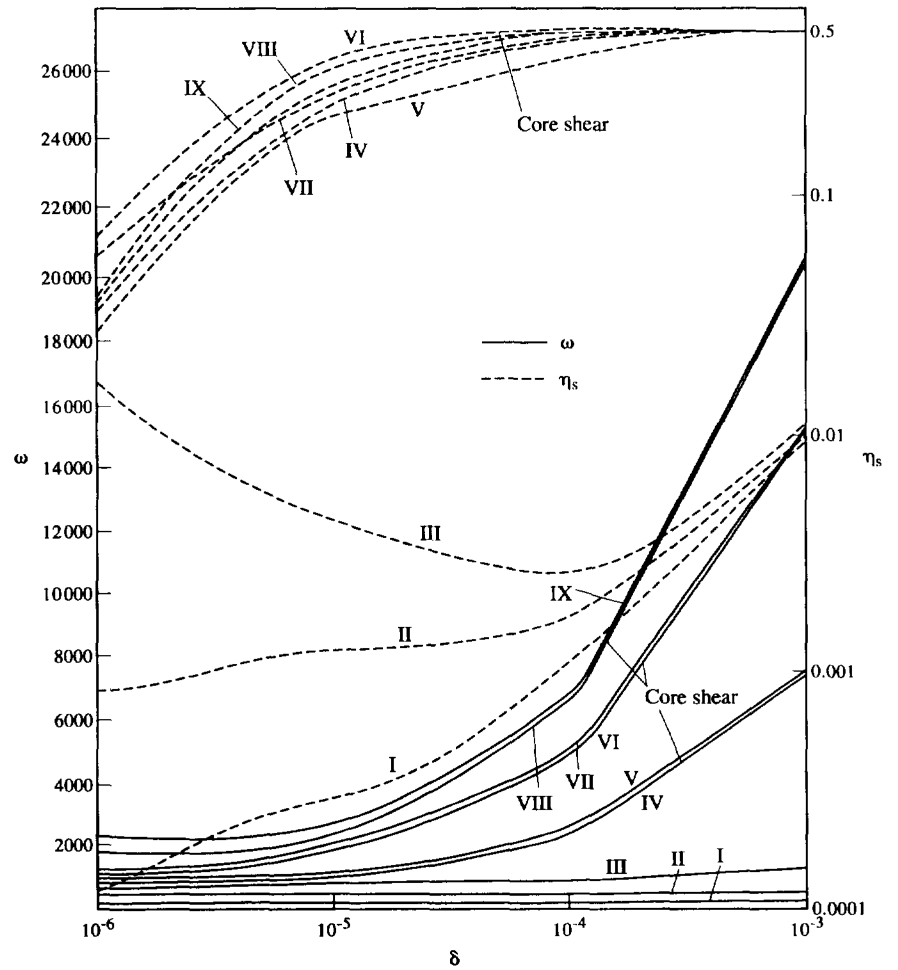

Fig. 8. Variation of ω and η_s with δ for antisymmetric vibration of a clamped–clamped seven-layered conical shell.

(f) Effect of V *on multilayered conical shells with free edges*

Figures 18–20 show the variation of ω and η_s with thickness ratio parameter V.

ω for Mode I for the multilayered shell has no appreciable change for the chosen range of V. ω for Modes II and III for the multilayered shell increases in the lower range of V and a decrease is observed with higher values of V. This trend of increase/decrease is insignificant when there are more layers in the shell. η_s for these modes varies with V in the same fashion as has been observed for simply supported/clamped–clamped shells.

ω and η_s for meridional and circumferential core shear modes have the same trend of variation with V as has been noticed for the other two cases.

6. CONCLUSIONS

In the investigation of multilayered conical shells of constant total thickness with all the three end conditions: simply supported at both ends, clamped–clamped and free–free, it is observed that there is considerable increase in the system loss factor, particularly for Modes I, II and III in the lower range of shear parameter (i.e. $\delta \leqslant 10^{-5}$) when the number of layers in the shell is increased from three to five, but the increase is only marginal when the number of layers is further increased to seven. A remarkable increase in system loss factors for these modes is found for increasing values of δ. The system loss factor for higher-order modes, due to thickness shear of core, increases in the chosen range of δ and it also increases with the number of layers in the multilayered shell for a particular value of δ.

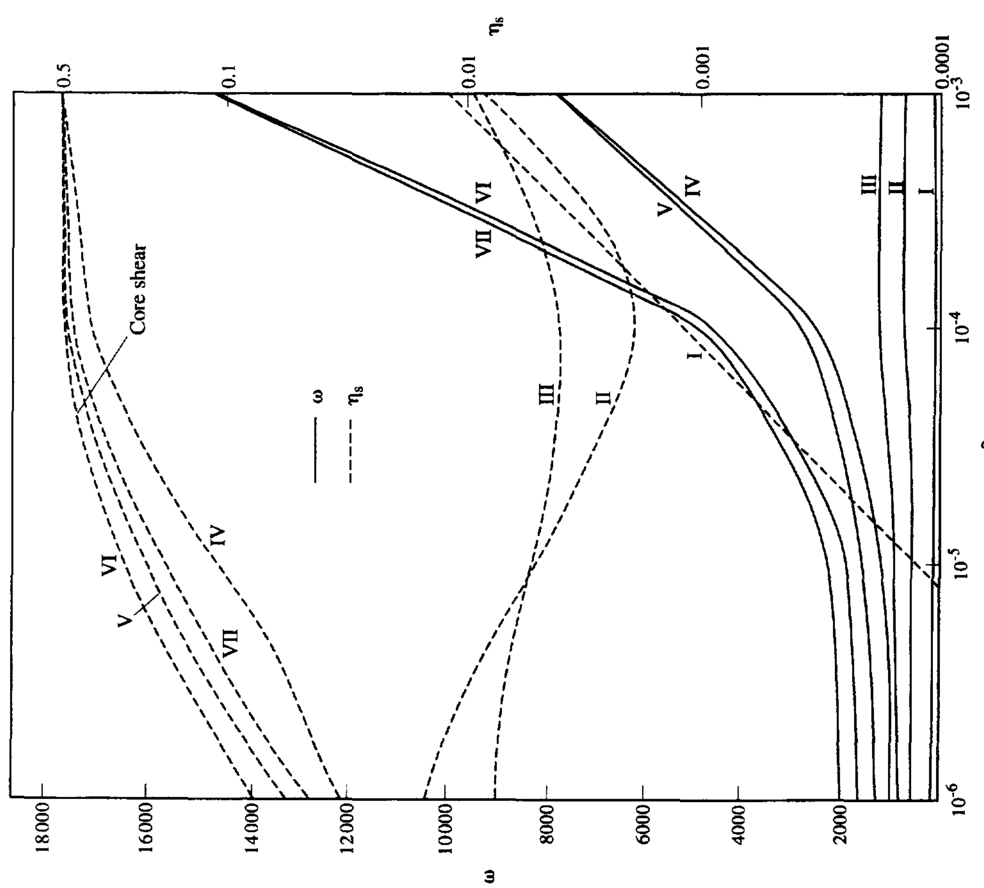

Fig. 10. Variation of ω and η_s with δ for antisymmetric vibration of a free–free five-layered conical shell.

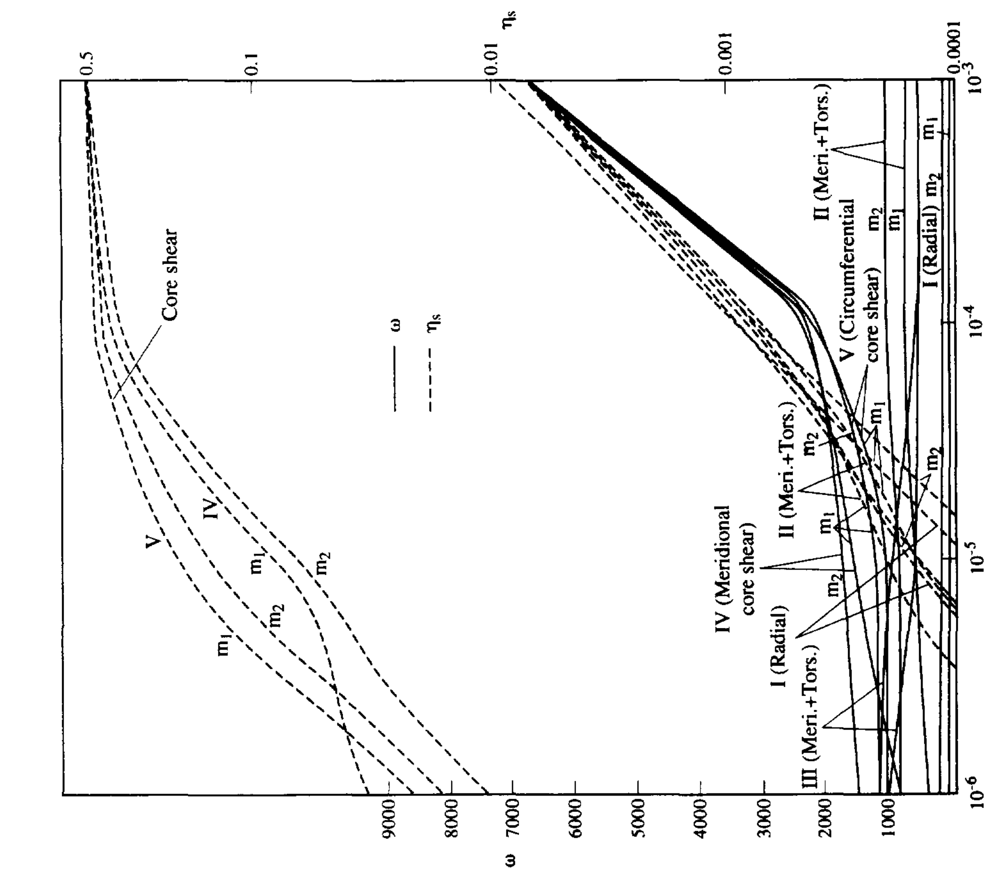

Fig. 9. Variation of ω and η_s with δ for antisymmetric vibration of a free–free three-layered conical shell.

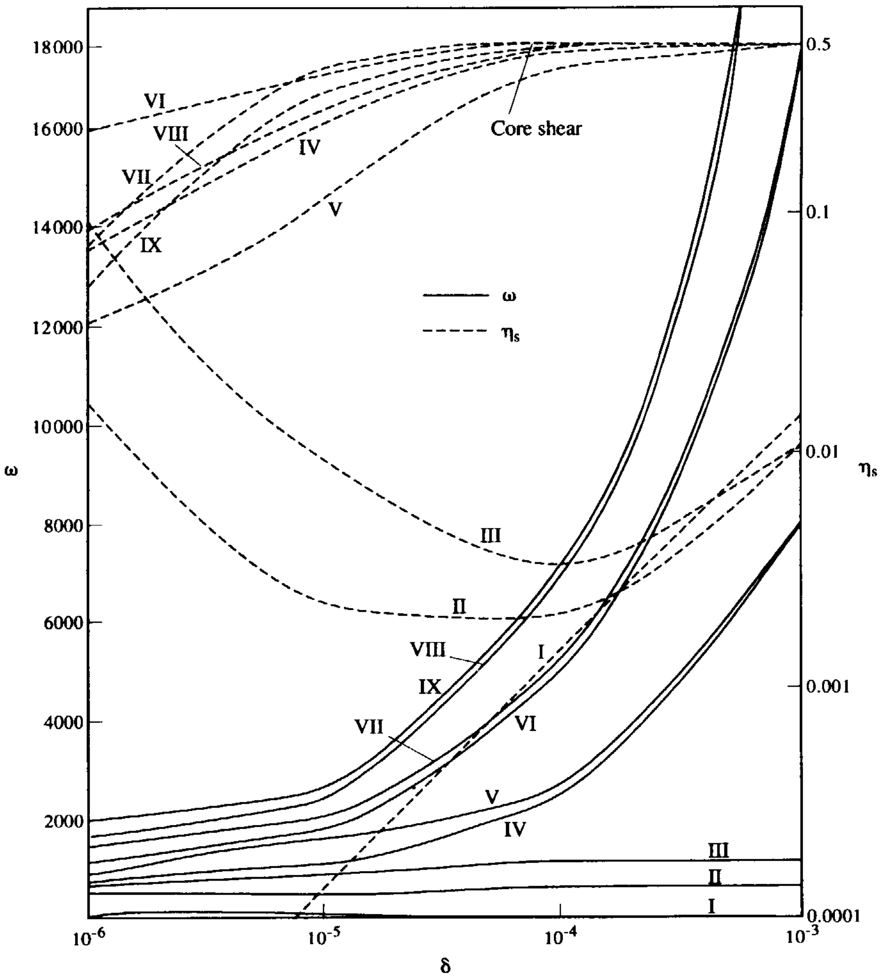

Fig. 11. Variation of ω and η_s with δ for antisymmetric vibration of a free–free seven-layered conical shell.

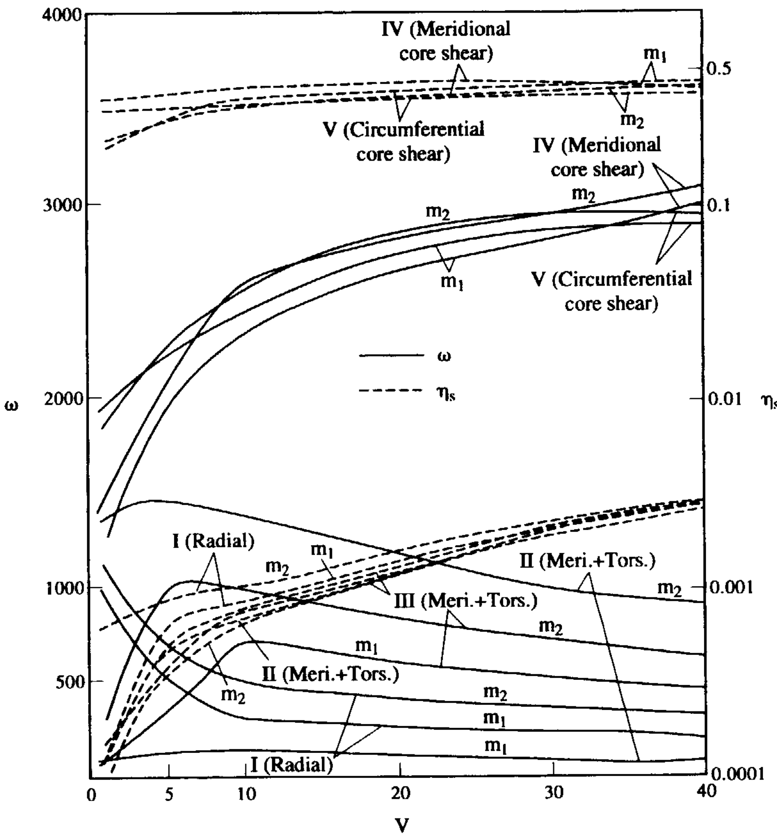

Fig. 12. Variation of ω and η_s with V for antisymmetric vibration of a simply supported three-layered conical shell.

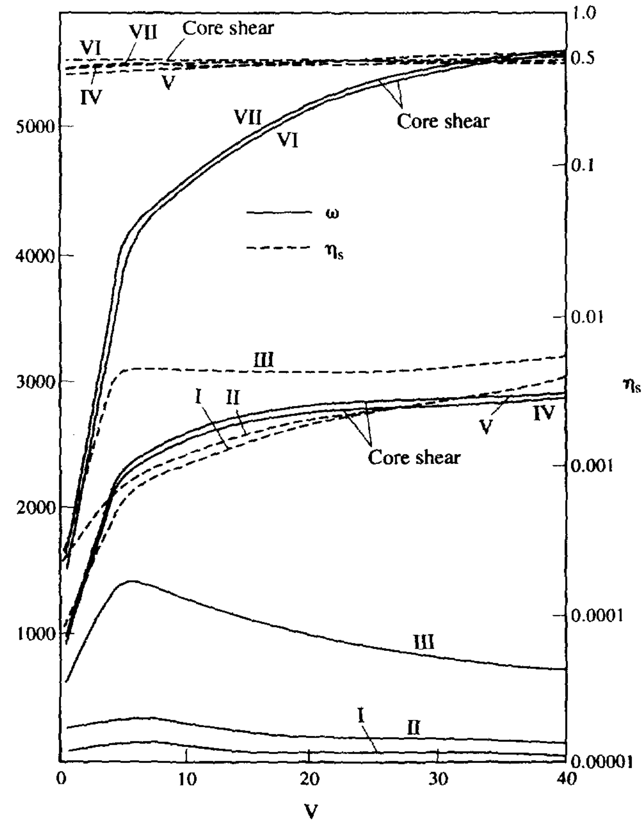

Fig. 13. Variation of ω and η_s with V for antisymmetric vibration of a simply supported five-layered conical shell.

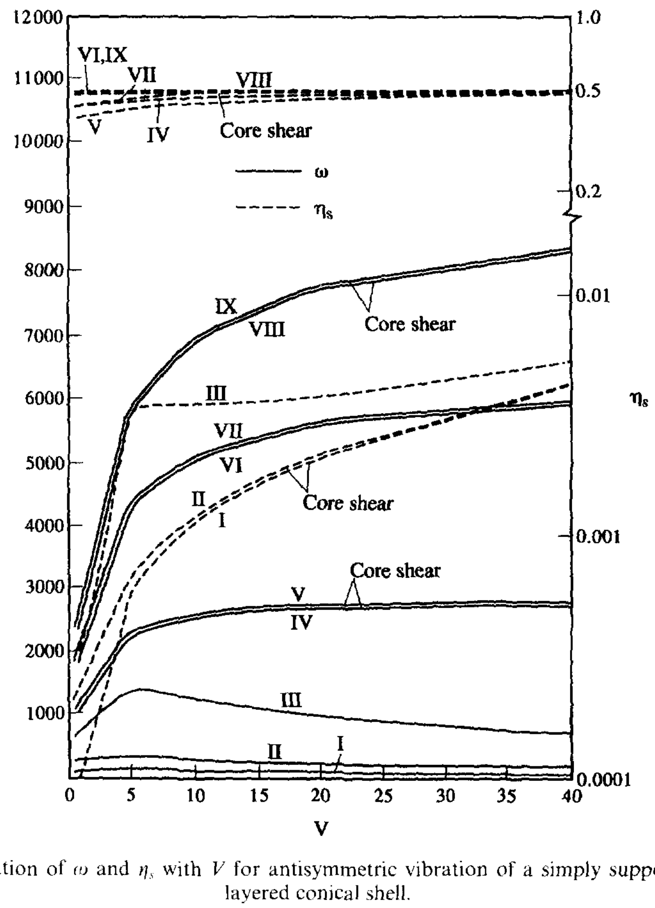

Fig. 14. Variation of ω and η_s with V for antisymmetric vibration of a simply supported seven-layered conical shell.

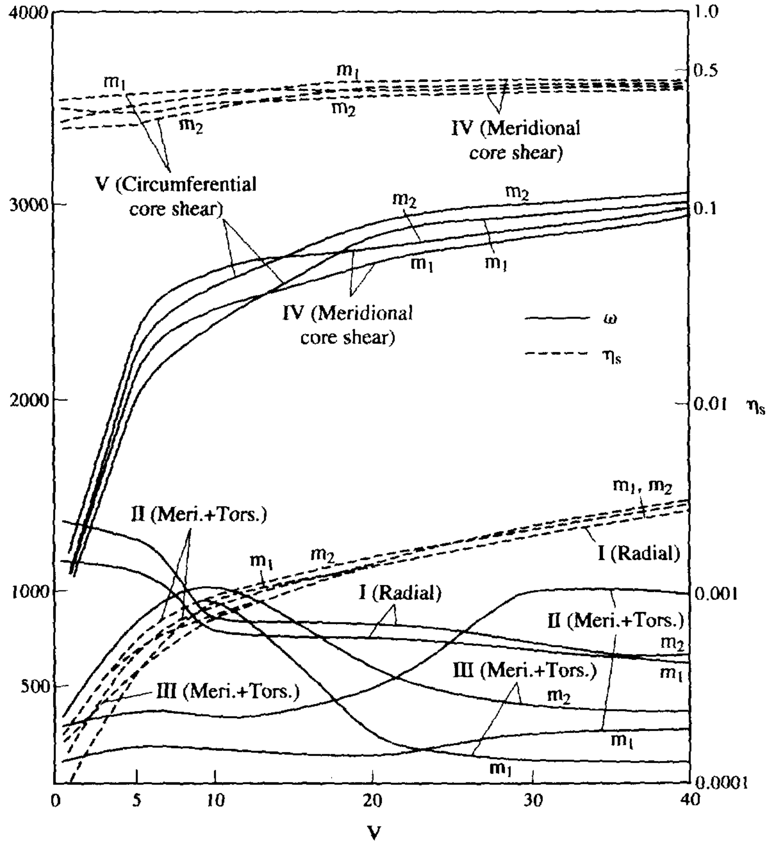

Fig. 15. Variation of ω and η_s with V for antisymmetric vibration of a clamped–clamped three-layered conical shell.

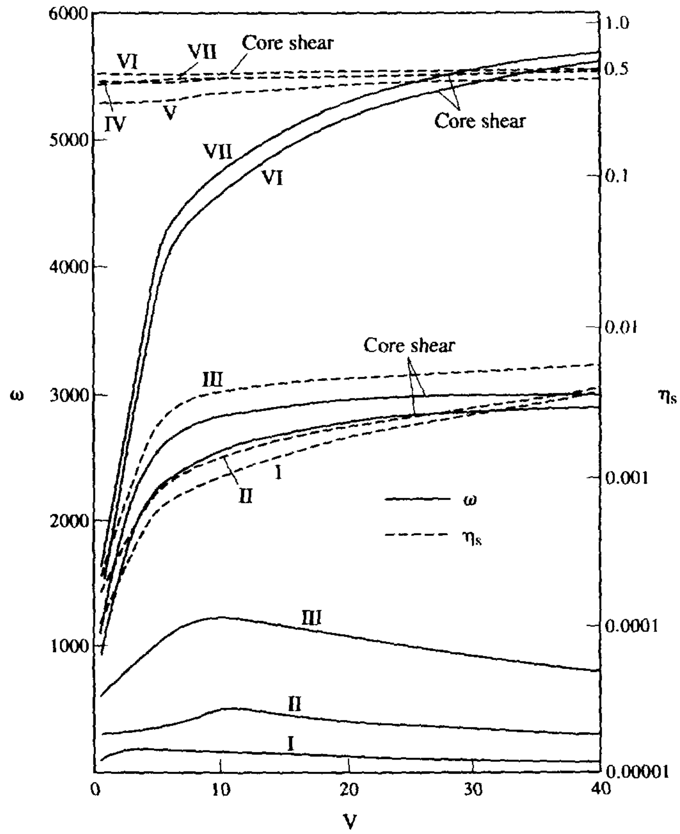

Fig. 16. Variation of ω and η_s with V for antisymmetric vibration of a clamped–clamped five-layered conical shell.

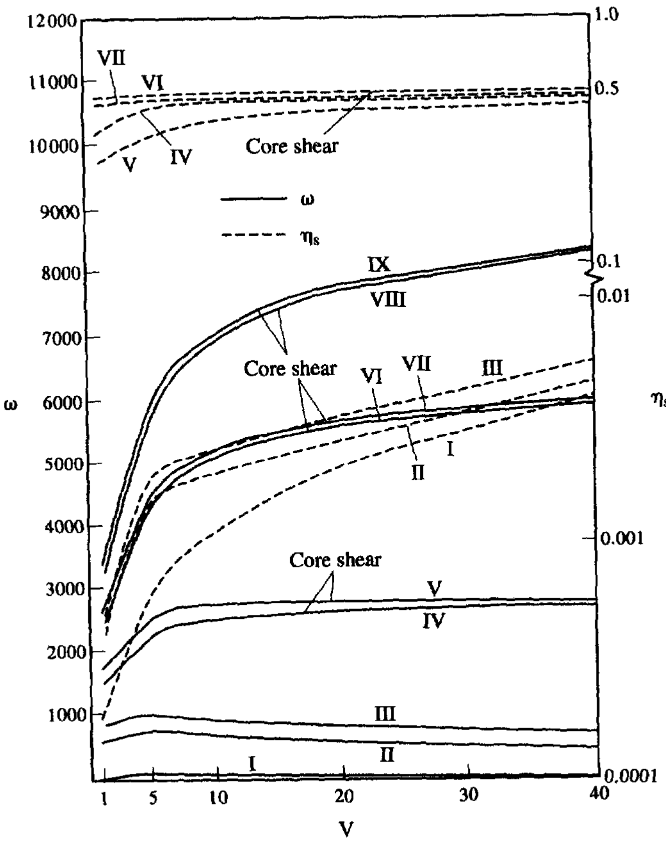

Fig. 17. Variation of ω and η_s with V for antisymmetric vibration of a clamped–clamped seven-layered conical shell.

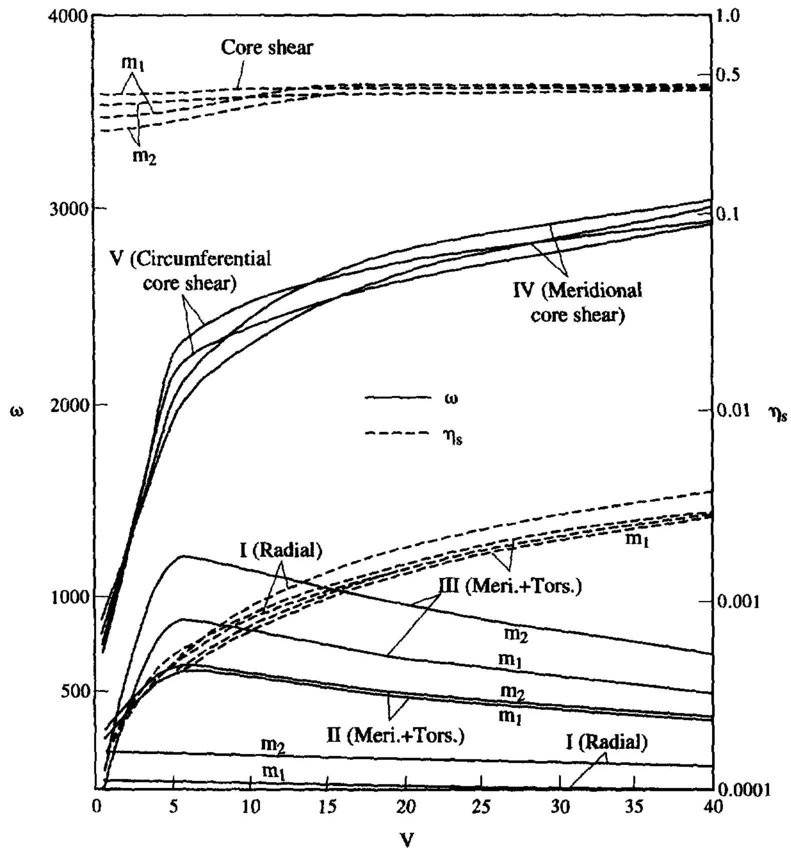

Fig. 18. Variation of ω and η_s with V for antisymmetric vibration of a free–free three-layered conical shell.

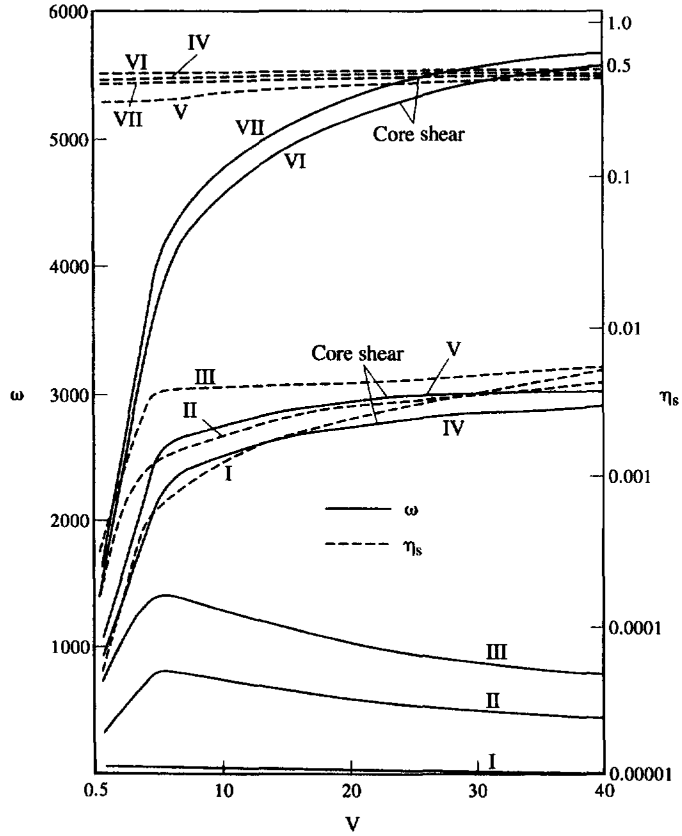

Fig. 19. Variation of ω and η_s with V for antisymmetric vibration of a free–free five-layered conical shell.

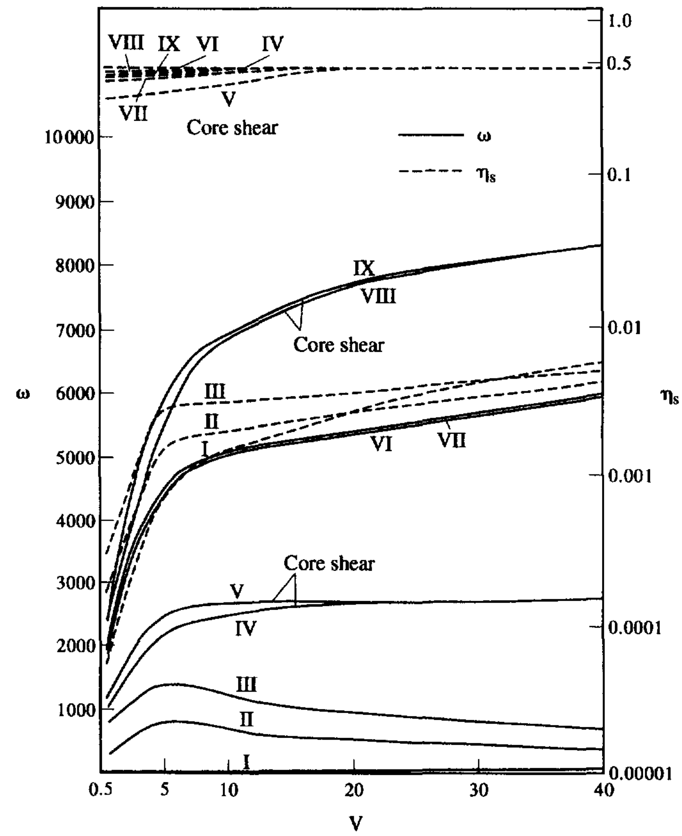

Fig. 20. Variation of ω and η_s with V for antisymmetric vibration of a free–free seven-layered conical shell.

Uniformly high values of the system loss factor for all families of modes of vibration for multilayered conical shells, for all the three boundary conditions, are obtained for $V > 10$, i.e. for thicker shells. The system loss factor is observed to be more for thickness core shear modes with a higher number of layers in the multilayered shell for a particular value of V.

In the present work, it is found that an increase in the number of layers increases the maximum obtainable system loss factor for most of the modes of vibration, with proper selection of the shear and thickness ratio parameters. For a uniformly high order of the system loss factor for all families of modes, one should choose higher values of δ, V, and more layers in the multilayered shell.

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APPENDIX

Strain energy

$$U = \frac{1}{2}\sum_{i=1}^{N}\int_x\int_\phi \frac{1}{t_i^2}\Bigg((Q_{11})_i\left[u_{i,x}^2\left(a_{i1}\frac{t_i^2}{4}+a_{i3}-a_{i2}t_i\right)+u_{i+1,x}^2\left(a_{i1}\frac{t_i^2}{4}+a_{i3}+a_{i2}t_i\right)+2u_{i,x}u_{i+1,x}\left(a_{i1}\frac{t_i^2}{4}-a_{i3}\right)\right]$$

$$+2(Q_{12})_i\left\{\left(a_{i7}\frac{t_i^2}{4}+a_{i8}\right)[u_{i,x}(v_{i,\phi}+u_i\sin\alpha)+u_{i+1,x}(v_{i+1,\phi}+u_{i+1}\sin\alpha)]+\left(a_{i7}\frac{t_i^2}{4}-a_{i8}\right)\right.$$

$$\left.\times[u_{i,x}(v_{i+1,\phi}+u_{i+1}\sin\alpha)+u_{i+1,x}(v_{i,\phi}+u_i\sin\alpha)]+\left(a_{i7}\frac{t_i^2}{2}\right)w\cos\alpha(u_{i,x}+u_{i+1,x})\right\}$$

$$+(Q_{22})_i\left[\left(a_{i4}\frac{t_i^2}{4}+a_{i6}-a_{i5}t_i\right)(v_{i,\phi}^2+u_i^2\sin^2\alpha+2v_{i,\phi}u_i\sin\alpha)+\left(a_{i4}\frac{t_i^2}{4}+a_{i6}+a_{i5}t_i\right)(v_{i+1,\phi}^2+u_{i+1}^2\sin^2\alpha\right.$$

$$+2v_{i+1,\phi}u_{i+1}\sin\alpha)+\left(a_{i4}\frac{t_i^2}{4}-a_{i6}\right)2(v_{i,\phi}v_{i+1,\phi}+v_{i,\phi}u_{i+1}\sin\alpha+v_{i+1,\phi}u_i\sin\alpha+u_iu_{i+1}\sin^2\alpha)$$

$$\left.+\left(a_{i4}\frac{t_i^2}{2}+a_{i5}t_i\right)2w\cos\alpha(v_{i+1,\phi}+u_{i+1}\sin\alpha)+\left(a_{i4}\frac{t_i^2}{2}-a_{i5}t_i\right)2w\cos\alpha(v_{i,\phi}+u_i\sin\alpha)+a_{i4}t_i^2w^2\cos^2\alpha\right]$$

$$
\begin{aligned}
&+(Q_{66})_i\left[\left(a_{i4}\frac{t_i^2}{4}+a_{i6}-a_{i5}t_i\right)(u_{i,\phi}^2+v_i^2\sin^2\alpha-2u_{i,\phi}v_i\sin\alpha)+\left(a_{i4}\frac{t_i^2}{4}+a_{i6}+a_{i5}t_i\right)(u_{i+1,\phi}^2+v_{i+1}^2\sin^2\alpha\right.\\
&-2u_{i+1,\phi}v_{i+1}\sin\alpha)+\left(a_{i4}\frac{t_i^2}{4}-a_{i6}\right)2(u_{i,\phi}u_{i-1,\phi}-u_{i,\phi}v_{i+1}\sin\alpha-u_{i+1,\phi}v_i\sin\alpha+v_iv_{i+1}\sin^2\alpha)\\
&+\left(a_{i1}\frac{t_i^2}{4}+a_{i3}-a_{i2}t_i\right)v_{i,x}^2+\left(a_{i1}\frac{t_i^2}{4}+a_{i3}+a_{i2}t_i\right)v_{i+1,x}^2+2v_{i,x}v_{i+1,x}\left(a_{i1}\frac{t_i^2}{4}-a_{i3}\right)+\left(a_{i7}\frac{t_i^2}{4}+a_{i8}\right)\\
&\times 2(v_{i,x}u_{i,\phi}-v_{i,x}v_i\sin\alpha+v_{i+1,x}u_{i+1,\phi}-v_{i+1,x}v_{i+1}\sin\alpha)+\left(a_{i7}\frac{t_i^2}{4}-a_{i8}\right)2(v_{i,x}u_{i+1,\phi}-v_{i,x}v_{i+1}\sin\alpha\\
&\left.+v_{i+1,x}u_{i,\phi}-v_{i+1,x}v_i\sin\alpha)\right]+(C_{44})_i\left[a_{i4}t_i^2w_{,\phi}^2+\left(a_{i4}\frac{t_i^2}{4}+a_{i6}-a_{i5}t_i\right)v_i^2\cos^2\alpha+\left(a_{i4}\frac{t_i^2}{4}+a_{i6}+a_{i5}\frac{t_i}{2}\right)\right.\\
&\times v_{i-1}^2\cos^2\alpha-2v_iw_{,\phi}\cos\alpha\left(a_{i4}\frac{t_i^2}{2}-a_{i5}t_i\right)+2v_iv_{i+1}\cos^2\alpha\left(a_{i4}\frac{t_i}{4}-a_{i6}\right)-2v_{i+1}w_{,\phi}\cos\alpha\left(a_{i4}\frac{t_i^2}{2}+a_{i5}t_i\right)\\
&\left.+a_{i1}(v_{i+1}^2+v_i^2-2v_iv_{i+1})+2a_{i7}t_iw_{,\phi}(v_{i+1}-v_i)+a_{i7}t_i\cos\alpha(v_i^2-v_{i+1}^2)\right]\\
&+(C_{55})_i[a_{i1}(t_i^2w_{,x}^2+u_{i-1}^2+u_i^2+2u_{i+1}w_{,x}t_i-2u_iu_{i+1}-2u_iw_{,x}t_i)]\,d\phi\,dx
\end{aligned}
\tag{A1}
$$

where

$$
\begin{aligned}
a_{i1} &= \int_{-t_i/2}^{t_i/2}\zeta_i^{-1}\,dz_i=(R_{0i}+x\sin\alpha)t_i\\
a_{i2} &= \int_{-t_i/2}^{t_i/2}z_i\zeta_i^{-1}\,dz_i=\frac{t_i^3}{12}\cos\alpha\\
a_{i3} &= \int_{-t_i/2}^{t_i/2}z_i^2\zeta_i^{-1}\,dz_i=\frac{t_i^3}{12}(R_{0i}+x\sin\alpha)\\
a_{i4} &= \int_{-t_i/2}^{t_i/2}\zeta_i\,dz_i=\frac{t_i}{(R_{0i}+x\sin\alpha)}+\frac{t_i^3\cos^2\alpha}{12(R_{0i}+x\sin\alpha)^3}\\
a_{i5} &= \int_{-t_i/2}^{t_i/2}z_i\zeta_i\,dz_i=-\frac{t_i^3\cos\alpha}{12(R_{0i}+x\sin\alpha)^2}\\
a_{i6} &= \int_{-t_i/2}^{t_i/2}z_i^2\zeta_i\,dz_i=\frac{t_i^3}{12(R_{0i}+x\sin\alpha)}\\
a_{i7} &= \int_{-t_i/2}^{t_i/2}dz_i=t_i\\
a_{i8} &= \int_{-t_i/2}^{t_i/2}z_i^2\,dz_i=\frac{t_i^3}{12}.
\end{aligned}
\tag{A2}
$$

Kinetic energy

$$
\begin{aligned}
T=\frac{1}{2}\sum_{i=1}^{N}\int_x\int_\phi\Bigg\{&\rho_ia_{i1}\dot{w}^2+\rho_i\left[a_{i1}\left(\frac{\dot{u}_i+\dot{u}_{i+1}}{2}\right)^2+\frac{a_{i3}}{t_i^2}(\dot{u}_{i+1}-\dot{u}_i)^2+\frac{a_{i2}}{t_i}(\dot{u}_{i+1}^2-\dot{u}_i^2)\right]\\
&+\rho_i\left[a_{i1}\left(\frac{\dot{v}_i+\dot{v}_{i+1}}{2}\right)^2+\frac{a_{i3}}{t_i^2}(\dot{v}_{i+1}-\dot{v}_i)^2+\frac{a_{i2}}{t_i}(\dot{v}_{i+1}^2-\dot{v}_i^2)\right]\Bigg\}\,d\phi\,dx
\end{aligned}
\tag{A3}
$$

Governing equations of motion and boundary conditions for antisymmetric vibrations of multilayered conical shell

$$
\begin{aligned}
&\left[\frac{1}{3}\frac{t_i}{(R_{0i}+x\sin\alpha)}+\frac{1}{12}\frac{t_i^2\cos\alpha}{(R_{0i}+x\sin\alpha)^2}+\frac{1}{48}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right][(Q_{22})_i\sin\alpha(u_i\sin\alpha+v_{i,\phi})\\
&+(Q_{66})_i(v_{i,\phi}\sin\alpha-u_{i,\phi\phi})]+\left[\frac{1}{6}\frac{t_i}{(R_{0i}+x\sin\alpha)}+\frac{1}{48}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right][(Q_{22})_i\sin\alpha(u_{i+1}\sin\alpha+v_{i+1,\phi})\\
&+(Q_{66})_i(v_{i+1,\phi}\sin\alpha-u_{i+1,\phi\phi})]+\left[\frac{1}{2}\frac{t_i}{(R_{0i}+x\sin\alpha)}+\frac{1}{12}\frac{t_i^2\cos\alpha}{(R_{0i}+x\sin\alpha)^2}+\frac{1}{24}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right]
\end{aligned}
$$

$$\times (Q_{22})_i w \sin\alpha\cos\alpha + (C_{55})_i (R_{0i}+x\sin\alpha)\left(\frac{1}{t_i}u_i - \frac{1}{t_i}u_{i+1} - w_{,x}\right) - \tfrac{1}{3}t_i(Q_{11})_i[u_{i,x}\sin\alpha$$

$$+ (R_{0i}+x\sin\alpha - \tfrac{1}{4}t_i\cos\alpha)u_{i,xx}] - \tfrac{1}{6}t_i(Q_{11})_i[u_{i+1,x}\sin\alpha + (R_{0i}+x\sin\alpha)u_{i+1,xx}] - \tfrac{1}{3}t_i(Q_{12})_i$$

$$\times (v_{i,\phi x} + \tfrac{1}{2}v_{i+1,\phi x}) - \tfrac{1}{2}t_i(Q_{12})_i w_{,x}\cos\alpha - \tfrac{1}{3}t_i(Q_{66})_i(v_{i,x\phi} + \tfrac{1}{2}v_{i+1,x\phi})$$

$$+\left[\frac{1}{3}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)} - \frac{1}{12}\frac{t_{i-1}^2\cos\alpha}{(R_{0i-1}+x\sin\alpha)^2} + \frac{1}{48}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right][(Q_{22})_{i-1}\sin\alpha(u_i\sin\alpha + v_{i,\phi})$$

$$+ (Q_{66})_{i-1}(v_{i,\phi}\sin\alpha - u_{i,\phi\phi})] + \left[\frac{1}{6}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)} + \frac{1}{48}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right]$$

$$\times [(Q_{22})_{i-1}\sin\alpha(v_{i-1,\phi} + u_{i-1}\sin\alpha) + (Q_{66})_{i-1}(v_{i-1,\phi}\sin\alpha - u_{i-1,\phi\phi})] + \left[\frac{1}{2}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)}\right.$$

$$\left. - \frac{1}{12}\frac{t_{i-1}^2\cos\alpha}{(R_{0i-1}+x\sin\alpha)^2} + \frac{1}{24}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right](Q_{22})_{i-1}w\sin\alpha\cos\alpha + (C_{55})_{i-1}(R_{0i-1}+x\sin\alpha)$$

$$\times\left(\frac{1}{t_{i-1}}u_i - \frac{1}{t_{i-1}}u_{i-1} + w_{,x}\right) - \tfrac{1}{3}t_{i-1}(Q_{11})_{i-1}[u_{i,x}\sin\alpha + (R_{0i-1}+x\sin\alpha + \tfrac{1}{4}t_{i-1}\cos\alpha)u_{i,xx}]$$

$$-\tfrac{1}{6}t_{i-1}(Q_{11})_{i-1}[u_{i-1,x}\sin\alpha + (R_{0i-1}+x\sin\alpha)u_{i-1,xx}] - \tfrac{1}{3}t_{i-1}(Q_{12})_{i-1}(v_{i,\phi x} + \tfrac{1}{2}v_{i-1,\phi x})$$

$$-\tfrac{1}{2}t_{i-1}(Q_{12})_{i-1}w_{,x}\cos\alpha - \tfrac{1}{3}t_{i-1}(Q_{66})_{i-1}(v_{i,x\phi} + \tfrac{1}{2}v_{i-1,x\phi}) + \tfrac{1}{3}\rho_i t_i(R_{0i}+x\sin\alpha)$$

$$\times(\ddot{u}_i + \tfrac{1}{2}\ddot{u}_{i-1}) + \tfrac{1}{3}\rho_{i-1}t_{i-1}(R_{0i-1}+x\sin\alpha)(\ddot{u}_i + \tfrac{1}{2}\ddot{u}_{i-1}) - \tfrac{1}{12}(\rho_i t_i^2 - \rho_{i-1}t_{i-1}^2)\ddot{u}_i = 0 \qquad \text{(A4)}$$

[for $i = 1, 2, 3, \ldots, (N+1)$, these are $(N+1)$ equations].

$$\left[\frac{1}{3}\frac{t_i}{(R_{0i}+x\sin\alpha)} + \frac{1}{12}\frac{t_i^2\cos\alpha}{(R_{0i}+x\sin\alpha)^2} + \frac{1}{48}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right][(Q_{66})_i v_i\sin^2\alpha - (Q_{66})_i u_{i,\phi}\sin\alpha$$

$$+ (C_{44})_i v_i\cos^2\alpha - (Q_{22})_i v_{i,\phi\phi} - (Q_{22})_i u_{i,\phi}\sin\alpha] + \left[\frac{1}{6}\frac{t_i}{(R_{0i}+x\sin\alpha)} + \frac{1}{48}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right]$$

$$\times[(Q_{66})_i v_{i+1}\sin^2\alpha - (Q_{66})_i u_{i+1,\phi}\sin\alpha + (C_{44})_i v_{i-1}\cos^2\alpha - (Q_{22})_i v_{i+1,\phi\phi} - (Q_{22})_i u_{i+1,\phi}\sin\alpha]$$

$$-\tfrac{1}{3}t_i(Q_{66})_i\sin\alpha(v_{i,x} + \tfrac{1}{2}v_{i+1,x}) - \left[\frac{1}{2}\frac{t_i}{(R_{0i}+x\sin\alpha)} + \frac{1}{12}\frac{t_i^2\cos\alpha}{(R_{0i}+x\sin\alpha)^2} + \frac{1}{24}\frac{t_i^3\cos^2\alpha}{(R_{0i}+x\sin\alpha)^3}\right]$$

$$\times[(C_{44})_i + (Q_{22})_i]w_{,\phi}\cos\alpha + \frac{1}{t_i}(C_{44})_i(R_{0i}+x\sin\alpha)(v_i - v_{i-1}) + (C_{44})_i(v_i\cos\alpha - w_{,\phi})$$

$$-\tfrac{1}{3}t_i(Q_{66})_i(R_{0i}+x\sin\alpha - \tfrac{1}{4}t_i\cos\alpha)v_{i,xx} - \tfrac{1}{6}t_i(Q_{66})_i(R_{0i}+x\sin\alpha)v_{i-1,xx} - \tfrac{1}{3}t_i(Q_{66})_i$$

$$\times(u_{i,\phi x} + \tfrac{1}{2}u_{i-1,\phi x}) - \tfrac{1}{3}t_i(Q_{12})_i(u_{i,x\phi} + \tfrac{1}{2}u_{i+1,x\phi}) + \left[\frac{1}{3}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)} - \frac{1}{12}\frac{t_{i-1}^2\cos\alpha}{(R_{0i-1}+x\sin\alpha)^2}\right.$$

$$\left. + \frac{1}{48}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right][(Q_{66})_{i-1}v_i\sin^2\alpha - (Q_{66})_{i-1}u_{i,\phi}\sin\alpha + (C_{44})_{i-1}v_i\cos^2\alpha - (Q_{22})_{i-1}v_{i,\phi\phi} - (Q_{22})_{i-1}$$

$$\times u_{i,\phi}\sin\alpha] + \left[\frac{1}{6}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)} + \frac{1}{48}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right][(Q_{66})_{i-1}v_{i-1}\sin^2\alpha - (Q_{66})_{i-1}u_{i-1,\phi}\sin\alpha$$

$$-(Q_{22})_{i-1}v_{i-1,\phi\phi} - (Q_{22})_{i-1}u_{i-1,\phi}\sin\alpha + (C_{44})_{i-1}v_{i-1}\cos^2\alpha] - \left[\frac{1}{2}\frac{t_{i-1}}{(R_{0i-1}+x\sin\alpha)} - \frac{1}{12}\frac{t_{i-1}^2\cos\alpha}{(R_{0i-1}+x\sin\alpha)^2}\right.$$

$$\left. + \frac{1}{48}\frac{t_{i-1}^3\cos^2\alpha}{(R_{0i-1}+x\sin\alpha)^3}\right]w_{,\phi}\cos\alpha[(C_{44})_{i-1} + (Q_{22})_{i-1}] - \tfrac{1}{3}t_{i-1}(Q_{66})_{i-1}(R_{0i-1}+x\sin\alpha + \tfrac{1}{4}t_{i-1}\cos\alpha)v_{i,xx}$$

$$-\tfrac{1}{6}t_{i-1}(Q_{66})_{i-1}(R_{0i-1}+x\sin\alpha)v_{i-1,xx} + \frac{1}{t_{i-1}}(C_{44})_{i-1}(R_{0i-1}+x\sin\alpha)(v_i - v_{i-1}) + (C_{44})_{i-1}(w_{,\phi}$$

$$-v_i\cos\alpha) - \tfrac{1}{3}t_{i-1}(Q_{66})_{i-1}(u_{i,\phi x} + \tfrac{1}{2}u_{i-1,\phi x}) - \tfrac{1}{3}t_{i-1}(Q_{12})_{i-1}(u_{i,x\phi} + \tfrac{1}{2}u_{i-1,x\phi}) - \tfrac{1}{3}t_{i-1}(Q_{66})_{i-1}\sin\alpha$$

$$\times(v_{i,x} + \tfrac{1}{2}v_{i-1,x}) + \tfrac{1}{3}\rho_i t_i(R_{0i}+x\sin\alpha)(\ddot{v}_i + \tfrac{1}{2}\ddot{v}_{i-1}) + \tfrac{1}{3}\rho_{i-1}t_{i-1}(R_{0i-1}+x\sin\alpha)$$

$$\times(\ddot{v}_i + \tfrac{1}{2}\ddot{v}_{i-1}) - \tfrac{1}{12}(\rho_i t_i^2 - \rho_{i-1}t_{i-1}^2)\ddot{v}_i\cos\alpha = 0 \qquad \text{(A5)}$$

[for $i = 1, 2, 3, \ldots, (N+1)$, these are $(N+1)$ equations], and

$$
\begin{aligned}
\sum_{i=1}^{N} \Bigg\{ & \tfrac{1}{2} t_i (Q_{12})_i \cos\alpha (u_{i,x} + u_{i+1,x}) + \left[\frac{t_i}{(R_{0i} + x\sin\alpha)} + \frac{1}{12} \frac{t_i^3 \cos^2\alpha}{(R_{0i} + x\sin\alpha)^3} \right] [(Q_{22})_i w \cos^2\alpha - (C_{44})_i w_{,\phi\phi}] \\
& + \left[\frac{1}{2} \frac{t_i}{(R_{0i} + x\sin\alpha)} - \frac{1}{12} \frac{t_i^2 \cos\alpha}{(R_{0i} + x\sin\alpha)^2} + \frac{1}{24} \frac{t_i^3 \cos^2\alpha}{(R_{0i} + x\sin\alpha)^3} \right] [(Q_{22})_i u_{i+1} \sin\alpha\cos\alpha + (Q_{22})_i v_{i+1,\phi} \cos\alpha \\
& + (C_{44})_i v_{i+1,\phi} \cos\alpha] + \left[\frac{1}{2} \frac{t_i}{(R_{0i} + x\sin\alpha)} + \frac{1}{12} \frac{t_i^2 \cos\alpha}{(R_{0i} + x\sin\alpha)^2} + \frac{1}{24} \frac{t_i^3 \cos^2\alpha}{(R_{0i} + x\sin\alpha)^3} \right] [(Q_{22})_i u_i \sin\alpha\cos\alpha \\
& + (Q_{22})_i v_{i,\phi} \cos\alpha + (C_{44})_i v_{i,\phi} \cos\alpha] - (C_{55})_i t_i [w_{,x} \sin\alpha + (R_{0i} + x\sin\alpha) w_{,xx}] - (C_{55})_i [u_{i+1} \sin\alpha \\
& + (R_{0i} + x\sin\alpha) u_{i+1,x}] + (C_{55})_i [u_i \sin\alpha + (R_{0i} + x\sin\alpha) u_{i,x}] + (C_{44})_i (v_{,\phi} - v_{i-1,\phi}) \\
& + \rho_i t_i (R_{0i} + x\sin\alpha) \ddot{w} + f(x,\phi) g(t) \Bigg\} = 0.
\end{aligned}
\tag{A6}
$$

The boundary conditions obtained at $x = 0$ and $x = L$ are either $u_i = 0$ or:

$$
\begin{aligned}
& \tfrac{1}{3} t_i (Q_{11})_i (R_{0i} + x\sin\alpha - \tfrac{1}{4} t_i \cos\alpha) u_{i,x} + \tfrac{1}{6} t_i (Q_{11})_i (R_{0i} + x\sin\alpha) u_{i+1,x} \\
& \quad + \tfrac{1}{3} t_i (Q_{12})_i (v_{i,\phi} + u_i \sin\alpha + \tfrac{1}{2} v_{i+1,\phi} + \tfrac{1}{2} u_{i+1} \sin\alpha + \tfrac{3}{2} w\cos\alpha) + \tfrac{1}{3} t_{i-1} (Q_{11})_{i-1} (R_{0i-1} \\
& \quad + x\sin\alpha + \tfrac{1}{4} t_{i-1} \cos\alpha) u_{i,x} + \tfrac{1}{6} t_{i-1} (Q_{11})_{i-1} (R_{0i-1} + x\sin\alpha) u_{i-1,x} \\
& \quad + \tfrac{1}{3} t_{i-1} (Q_{12})_{i-1} (v_{i,\phi} + u_i \sin\alpha + \tfrac{1}{2} v_{i-1,\phi} + \tfrac{1}{2} u_{i-1} \sin\alpha + \tfrac{3}{2} w\cos\alpha) = 0
\end{aligned}
\tag{A7}
$$

[for $i = 1, 2, 3, \ldots, (N+1)$, these are $(N+1)$ equations]

either $v_i = 0$ or:

$$
\begin{aligned}
& \tfrac{1}{3} t_i (Q_{66})_i (R_{0i} + x\sin\alpha - \tfrac{1}{4} t_i \cos\alpha) v_{i,x} + \tfrac{1}{6} t_i (Q_{66})_i (R_{0i} + x\sin\alpha) v_{i+1,x} + \tfrac{1}{3} t_i (Q_{66})_i (u_{i,\phi} - v_i \sin\alpha \\
& \qquad + \tfrac{1}{2} u_{i-1,\phi} - \tfrac{1}{2} v_{i+1} \sin\alpha) + \tfrac{1}{3} t_{i-1} (Q_{66})_{i-1} (R_{0i-1} + x\sin\alpha + \tfrac{1}{4} t_{i-1} \cos\alpha) v_{i,x} \\
& \quad + \tfrac{1}{6} t_{i-1} (Q_{66})_{i-1} (R_{0i-1} + x\sin\alpha) v_{i-1,x} + \tfrac{1}{3} t_{i-1} (Q_{66})_{i-1} (u_{i,\phi} - v_i \sin\alpha + \tfrac{1}{2} u_{i-1,\phi} - \tfrac{1}{2} v_{i-1} \sin\alpha) = 0
\end{aligned}
\tag{A8}
$$

[for $i = 1, 2, 3, \ldots, (N+1)$, these are $(N+1)$ equations]

and either $w = 0$ or

$$
\sum_{i=1}^{N} [(C_{55})_i (R_{0i} + x\sin\alpha)(t_i w_{,x} + u_{i+1} - u_i)] = 0. \tag{A9}
$$